

# Orbifold Gromov-Witten Theory<sup>1</sup>

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## 1 Introduction

In 1985, Dixon, Harvey, Vafa and Witten considered string theory over orbifolds (arising as global quotients  $X/G$  by a finite group  $G$ )[DHVW]. Although an orbifold is a singular space, orbifold string theory is surprisingly a "smooth" string theory. Since then, orbifold string theory has become a rather important part of the landscape of string theory. Although orbifold string theory has been around for a while, it was apparently poorly explored in mathematics. For the last fifteen years, only a small piece of orbifold string theory concerning the orbifold Euler number has been studied in mathematics. This paper is one of our several efforts [CR1], [Ru2], [AR] to change the situation.

Even with a superficial understanding of orbifold string theory, it is obvious that the mathematics surrounding orbifold string theory is striking. In fact, it brings in much new mathematics unique to orbifolds. We believe that there is an emerging new topology and geometry of orbifolds. The core of this new geometry and topology is the concept of twisted sectors. Roughly speaking, the consistency of orbifold string theory requires that the string Hilbert space has to contain factors

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called twisted sectors. Twisted sectors can be viewed as the contribution from singularities. All other quantities such as correlation functions have to contain the contribution from the twisted sectors. In [CR1], we studied twisted sectors in the context of classical topology and obtained a new cohomology theory for orbifolds (orbifold cohomology). In this paper, we continue our work in quantum theory to construct an orbifold quantum cohomology. Recall that ordinary quantum cohomology is a deformation of ordinary cohomology. Orbifold quantum cohomology can be thought as a deformation of orbifold cohomology.

Another motivation of this paper comes from mirror symmetry. The current mirror symmetry is restricted to Calabi-Yau 3-folds only. The most of known Calabi-Yau 3-folds are so called crepant resolution of Calabi-Yau orbifolds. In higher dimension, there are still plenty of Calabi-Yau orbifolds. But they do not have crepant resolution in general. Therefore, there is no hope to consider mirror symmetry for smooth higher dimensional Calabi-Yau manifolds. We are forced to work with orbifolds! This paper can be viewed as the first step towards higher dimensional mirror symmetry.

The first step of our paper is to generalize the notion of stable map. This is a nontrivial step. Recall that orbifold is covered by orbifold charts of the form  $U/G$ , where  $U$  is a smooth manifold and  $G$  is a finite group acting on  $U$ . Let  $G_p$  be the subgroup of  $G$  fixing a point of the preimage of  $p$ .  $G_p$  is well-defined up to conjugation and is called the local group of  $p$ . The natural generalization of a smooth map in the orbifold category is the orbifold map  $f : X \rightarrow Y$ , where locally  $f_p : U_p/G_p \rightarrow V_{f(p)}/G_{f(p)}$  can be lifted to a map  $\tilde{f} : U_p \rightarrow V_{f(p)}$  and an injective homomorphism  $f_{\#} : G_p \rightarrow G_{f(p)}$ . Of course, the pair  $(\tilde{f}, f_{\#})$  is only defined up to conjugation of an element of  $G_{f(p)}$ . Suppose  $X$  is a marked orbifold Riemann surface with orbifold point  $(x_1, \dots, x_k)$ . For simplicity, we assume that all orbifold points are marked points. The local group of  $x_i$  is determined by an integer  $k_i$  representing the order of the local group. Then an orbifold map  $f : X \rightarrow Y$  is completely determined by the map itself and the conjugacy classes  $\mathbf{x} = ((g_1), \dots, (g_k))$  (We call it the orbifold type or twisted boundary condition of  $f$ ) for  $g_i \in G_{f(x_i)}$ . The problem is that such a straightforward generalization is wrong for the purposes of quantum cohomology, where we have to consider the deformation theory of such a map. Suppose that  $E \rightarrow Y$  is an orbifold bundle of  $Y$ . The critical issue is that the pull-back  $f^*E$  is not an orbifold bundle in general. If  $f^*E$  has an orbifold bundle structure, it often has more than one orbifold bundle structure. The critical step of this paper is to formulate the correct structure (*compatible system*) such that the usual deformation theory is possible. We do this in the appendix and call an orbifold map with such a structure a *good map*. This extra structure has a far-reaching influence on the structure of orbifold cohomology. Once we get over this conceptual hurdle, we define an orbifold stable map as nodal good map with the usual stability condition. Let  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  be the moduli space of orbifold stable maps of type  $\mathbf{x}$ . Then we prove

**Theorem A (Theorem 2.3.8):** *Suppose that  $X$  is either a symplectic orbifold with tamed almost complex structure or a projective orbifold. The moduli space of orbifold stable maps  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  is a compact metrizable space under a natural topology, whose “virtual dimension” is  $2d$ , where*

$$d = c_1(TX) \cdot A + (\dim_{\mathbf{C}} X - 3)(1 - g) + k - \iota(\mathbf{x}).$$

Here  $\iota(\mathbf{x}) := \sum_{i=1}^k \iota_{(g_i)}$  for  $\mathbf{x} = (X_{(g_1)}, \dots, X_{(g_k)})$  and the degree shifting number is defined in [CR1].

For any component  $\mathbf{x} = (X_{(g_1)}, \dots, X_{(g_k)})$ , there are  $k$  evaluation maps (cf. (3.5))

$$(1.1) \quad e_i : \overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x}) \rightarrow X_{(g_i)}, \quad i = 1, \dots, k.$$

There is a map

$$(1.2) \quad p :: \overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x}) \rightarrow \overline{\mathcal{M}}_{g,k}$$

defined by contracting the unstable components of the domain. For any set of cohomology classes  $\alpha_i \in H^{*-2\iota_{(g_i)}}(X_{(g_i)}; \mathbf{Q}) \subset H^*_{orb}(X; \mathbf{Q})$ ,  $i = 1, \dots, k$ ,  $K \in H^*(\overline{\mathcal{M}}_{g,k}, \mathbf{Q})$ , the orbifold Gromov-Witten invariant is defined as the pairing

$$(1.3) \quad \Psi_{(g,k,A,\mathbf{x})}^{X,J}(\mathcal{O}_{l_1}(\alpha_1), \dots, \mathcal{O}_{l_k}(\alpha_k)) = \int_{\overline{\mathcal{M}}_{g,k}(X,J,A,\mathbf{x})}^{vir} \prod_{i=1}^k c_1(L_i)^{l_i} e_i^* \alpha_i,$$

where  $L_i$  is the line bundle generated by cotangent space of the  $i$ -th marked point.

**Theorem B (Proposition 3.4.1):**

1.  $\Psi_{(g,k,A,\mathbf{x})}^{X,J}(K; \mathcal{O}_{l_1}(\alpha_1), \dots, \mathcal{O}_{l_k}(\alpha_k)) = 0$  unless  $\deg K + \sum_i (\deg_{orb}(\alpha_i) + l_i) = 2C_1(A) + 2n(3 - g) + 2k$ , where  $\deg_{orb}(\alpha_i)$  is orbifold degree of  $\alpha_i$  obtained after degree shifting.
2.  $\Psi_{(g,k,A,\mathbf{x})}^{X,J}(K; \mathcal{O}_{l_1}(\alpha_1), \dots, \mathcal{O}_{l_k}(\alpha_k))$  is independent of the choice of  $J$  and hence is an invariant of the symplectic structure.

**Theorem C (Theorem 3.4.2):**  $\Psi_{(g,k,A,\mathbf{x})}^{X,J}$  satisfy the same axioms as the ordinary GW-invariants except divisor axiom where we have to restrict ourself to divisor class in the nontwisted sector.

Similar to the smooth case, the genus zero orbifold GW-invariants can be used to define an orbifold quantum cohomology which deforms the orbifold cohomology constructed in [CR1].

**Theorem D (Theorem 3.4.3):** *The orbifold quantum product is associative.*

The moduli space of orbifold stable maps is constructed in section 2. In section 3, we carry out the construction of virtual cycles to define orbifold Gromov-Witten invariants. In this section, we adapted the technique from the smooth case developed by [FO], [LT], [Ru1], [S]. In particular, we follow the proof of [FO] closely. However, the analysis is more complicated because of the existence of singularities.

The results of this paper was announced in [CR2]. The second author would like to thank R. Dijkgraaf, E. Witten and E. Zaslow for many stimulating discussions about orbifold string theory. Orbifold stable maps have been studied independently in algebraic setting in [AV]. Both authors would like to thank D. Abramovich for clarifying the relation between our and their versions of orbifold stable maps.

## 2 Orbifold Stable Maps

### 2.1 Almost complex orbifolds and pseudo-holomorphic maps

We follow the notations from the appendix. We refer readers to the appendix for the definitions.

**Definition 2.1.1:** *An almost complex structure on an orbifold  $X$  is a  $C^\infty$  section  $J$  of the orbifold bundle  $\text{End}(TX)$  of endomorphisms of  $TX$  such that  $J^2 = -\text{Id}$ . The pair  $(X, J)$  is called an almost complex orbifold. A continuous map  $f : (X, J) \rightarrow (X', J')$  is said to be pseudo-holomorphic if there is a  $C^\infty$  map  $\tilde{f}$  lifting  $f$  such that  $d\tilde{f} \circ J = J' \circ d\tilde{f}$ .*

The next lemma is obvious, we leave the details to the reader.

**Lemma 2.1.2:** *Let  $(X, J)$  be an almost complex orbifold. Then  $J$  induces an almost complex structure on  $\widetilde{\Sigma X}$  for which the canonical resolution map  $\pi : \widetilde{\Sigma X} \rightarrow X$  is pseudo-holomorphic. In particular, the set of singularities  $\Sigma X$  is a pseudo-holomorphic subvariety in  $(X, J)$ .*

When the domain is 2-dimensional, it is convenient to use the following slightly different version of pseudo-holomorphic map.

**Definition 2.1.3:** *Let  $\Sigma$  be a Riemann surface with complex structure  $j$ , and  $(X, J)$  be an almost complex orbifold. A continuous map  $f : \Sigma \rightarrow X$  is called pseudo-holomorphic if for any point  $z_0 \in \Sigma$ , the following is true:*

1. *There is a disc neighborhood of  $z_0$  with a branched covering map  $br : z \rightarrow z^m$ . (Here  $m = 1$  is allowed.)*
2. *There is a local chart  $(V_{f(z_0)}, G_{f(z_0)}, \pi_{f(z_0)})$  of  $X$  at  $f(z_0)$  and a local lifting  $\tilde{f}_{z_0}$  of  $f$  in the sense that  $f \circ br = \pi_{f(z_0)} \circ \tilde{f}_{z_0}$ .*
3.  *$\tilde{f}_{z_0}$  is pseudo-holomorphic, i.e.,  $d\tilde{f}_{z_0} \circ j = J \circ d\tilde{f}_{z_0}$ .*

We shall be interested in moduli spaces of pseudo-holomorphic maps from a Riemann surface into an almost complex orbifold. The next lemma collects a few analytic properties of them which are very useful to keep in mind.

**Lemma 2.1.4:** *Let  $f : (\Sigma, j) \rightarrow (X, J)$  be a non-constant pseudo-holomorphic map such that  $f(\Sigma) \cap \Sigma X \neq \emptyset$ . Then there are two possibilities:*

1. *There are finitely many distinct points  $z_i \in \Sigma$ ,  $i = 1, \dots, k$ , such that  $f^{-1}(\Sigma X) = \{z_i\}$  and the image  $f(\Sigma)$  intersects  $\Sigma X$  at each  $f(z_i)$  with a finite order tangency. More precisely, suppose  $\tilde{f}_{z_i} : D \rightarrow V_{f(z_i)}$  is a local lifting of  $f$ ; then  $\tilde{f}_{z_i}(D)$  intersects with finite order tangency the preimage of  $\Sigma X$  in  $V_{f(z_i)}$ , which is a union of finitely many embedded pseudo-holomorphic submanifolds.*
2. *The image  $f(\Sigma)$  lies entirely in  $\Sigma X$ . In this case, there is a set  $\{z_1, \dots, z_k\} \subset \Sigma$  and a connected stratum of  $\Sigma X = \widetilde{\Sigma X}_{gen}$  containing  $f(\Sigma \setminus \{z_1, \dots, z_k\})$ . Therefore, after compactification,  $f$  is lifted to a pseudo-holomorphic map  $f_1 : \Sigma \rightarrow \widetilde{\Sigma X}$ , and  $f_1(\Sigma)$  intersects the singular set of  $\widetilde{\Sigma X}$  at finitely many isolated points with finite order tangency.*

**Proof:** Let  $z \in \Sigma$  be any point such that  $f(z) = p$  lies in  $\Sigma X$ . By Definition 2.1.3, there is a local lifting of  $f$ ,  $\tilde{f} : D \rightarrow V_p$ , satisfying  $d\tilde{f} \circ j = J \circ d\tilde{f}$ , where  $z$  and  $\tilde{f}(0)$  are the origins of  $D$  and  $V_p$ , respectively. The preimage of  $\Sigma X$  in  $V_p$  is a union of finitely many embedded pseudo-holomorphic submanifolds. Let  $W$  be any component; we can choose a complex coordinate system  $u_i$ ,  $i = 1, \dots, n$ , such that  $u_i(\tilde{f}(0)) = 0$  and  $u_i$ ,  $i = k+1, \dots, n$ , is a complex coordinate system for  $W$  near  $\tilde{f}(0)$ . Then the equation  $d\tilde{f} \circ j = J \circ d\tilde{f}$  is written as

$$\frac{\partial u_i}{\partial s} + J(u_1, \dots, u_n) \frac{\partial u_i}{\partial t} = 0, \quad i = 1, \dots, n,$$

where  $z = s + it$  is the complex coordinate on  $D$ . Let  $\Delta = \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}$  be the standard Laplacian on  $D$ ; then the above equation implies

$$\Delta u_i = (\partial_t J(u)) \partial_s u_i - (\partial_s J(u)) \partial_t u_i, \quad i = 1, \dots, n,$$

where  $u = (u_1, \dots, u_n)$ . On the other hand, if we write  $J(u)$  as a matrix

$$\begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

where  $A(u)$  is a  $k \times k$  matrix, then  $B(0, \dots, 0, u_{k+1}, \dots, u_n) = 0$  ( $W$  is  $J$ -pseudo-holomorphic), which implies that  $|\frac{\partial B}{\partial u_i}(u)| \leq C(|u_1| + \dots + |u_k|)$  for  $i = k+1, \dots, n$  with a constant  $C$  for all  $u$  with  $|u| \leq \delta_0$ . If we denote  $v = (u_1, \dots, u_k)$ , then there is a constant  $C'$  such that

$$|\Delta v(z)| \leq C'(|v| + |\partial_s v| + |\partial_t v|)$$

holds for all  $z \in D$  with  $|z| \leq \epsilon_0$  for some  $\epsilon_0$ . By Hartman-Wintner's lemma [HW],  $v$  is either identically zero or  $v(z) = az^m + O(|z|^{m+1})$  for some  $0 \neq a \in \mathbf{C}$ . Therefore we prove that either  $\tilde{f}(D)$  is entirely in the preimage of  $\Sigma X$  in  $V_p$ , or intersects it with finite order tangency.

Now suppose that  $f(\Sigma)$  is entirely in  $\Sigma X$ . Then by the canonical stratification of  $X$ , we have  $\Sigma X = \widetilde{\Sigma X}_{gen}$ . The points in  $\Sigma$  are divided into two groups,  $I$  and  $II$ , according to the following rule:  $z$  is in  $I$  if a neighborhood of  $z$  is mapped into the same stratum of the canonical stratification as  $z$  under  $f$ ;  $z$  is in  $II$  otherwise. Then it follows that  $II$  consists of only finitely many isolated points. This implies that the subset of  $\Sigma$  consisting of points in  $I$  is connected, so that its image under  $f$  lies in a connected open stratum of  $\widetilde{\Sigma X}_{gen}$ . After compactification, we obtain a unique pseudo-holomorphic map  $f_1 : \Sigma \rightarrow \widetilde{\Sigma X}$  such that  $\pi \circ f_1 = f$ , where  $\pi : \widetilde{\Sigma X} \rightarrow \Sigma X$  is the canonical resolution. Moreover,  $f_1$  intersects the singular set of  $\widetilde{\Sigma X}$  at finitely many points with finite order tangency.  $\square$

In Gromov-Witten theory, it is of primary importance that the moduli spaces under consideration admit certain compactifications. We shall consider two classes of almost complex orbifolds for which we can prove the compactness of moduli spaces. One of them is given in the following

**Definition 2.1.5:** A 2-form on an orbifold  $X$  is said to be non-degenerate if each local lifting is non-degenerate. A symplectic orbifold is an orbifold equipped with a closed, non-degenerate, 2-form  $\omega$ , usually denoted by  $(X, \omega)$ . An almost complex structure  $J$  on a symplectic orbifold  $(X, \omega)$  is said to be  $\omega$ -tamed if  $\omega(v, Jv) > 0$  for any nonzero  $v \in T_p X$ . It is called  $\omega$ -compatible if  $\omega(\cdot, J\cdot)$  is a Riemannian metric on  $X$ . The space of  $\omega$ -tamed almost complex structures on  $(X, \omega)$  is denoted by  $\mathcal{J}_\omega$ .

The same proof as in the smooth case yields

**Lemma 2.1.6:**  $\mathcal{J}_\omega$  is non-empty and contractible.

Another class of almost complex orbifolds comes from algebraic geometry. Suppose that  $(X, J)$  is an almost complex orbifold. If the local lifting of the almost complex structure  $J$  on each uniformizing system  $(V, G, \pi)$  is integrable, then  $(V, J)$  becomes a complex manifold and  $G$  acts on it holomorphically. The quotient space  $U = \pi(V)$  inherits an analytic structure such that the map  $\pi : V \rightarrow U$  is analytic. As a consequence,  $X$  is an analytic space. We call  $X$  a *complex orbifold*.

A continuous map  $f : (\Sigma, j) \rightarrow (X, J)$  from a Riemann surface into a complex orbifold  $(X, J)$  is called *analytic* if, for any local analytic function  $g$  on  $X$ , the pull-back  $g \circ f$  is holomorphic on  $\Sigma$ .

**Lemma 2.1.7:** A continuous map  $f : (\Sigma, j) \rightarrow (X, J)$  is analytic if and only if it is pseudo-holomorphic in the sense of Definition 2.1.3.

**Proof:** Suppose  $f$  is pseudo-holomorphic. Then at each point  $z_0 \in \Sigma$ , there is a disc neighborhood of  $z_0$  with a branched covering map  $br : z \rightarrow z^m$  (Here  $m = 1$  is allowed), a local chart

$(V_{f(z_0)}, G_{f(z_0)}, \pi_{f(z_0)})$  of  $X$  at  $f(z_0)$  and a local pseudo-holomorphic lifting  $\tilde{f}_{z_0}$  of  $f$  in the sense that  $f \circ br = \pi_{f(z_0)} \circ \tilde{f}_{z_0}$  and  $d\tilde{f}_{z_0} \circ j = J \circ d\tilde{f}_{z_0}$ . A local function  $g$  is analytic if and only if  $g \circ \pi_{f(z_0)}$  is holomorphic on  $V_{f(z_0)}$ . So if  $g$  is a local analytic function, then  $g \circ \pi_{f(z_0)} \circ \tilde{f}_{z_0}$  is holomorphic, which implies that  $g \circ f \circ br$  is holomorphic. So  $g \circ f$  is holomorphic, and hence  $f$  is analytic.

Suppose that  $f$  is analytic. For any connected component of  $\widetilde{\Sigma X}$ , its image under the canonical resolution map  $\pi : \widetilde{\Sigma X} \rightarrow X$  is an analytic subvariety. So the image of  $f$  is either contained in it or intersects it at finitely many points. So there is a connected component  $X_0$  of  $\widetilde{\Sigma X}$  such that  $f$  is lifted to an analytic map  $f_1 : \Sigma \rightarrow X_0$  with the property that for all  $z$  except finitely many  $z_1, \dots, z_k$ ,  $f_1(z)$  is in the set of regular points of  $X_0$ . It follows easily from this that  $f$  is pseudo-holomorphic.  $\square$

The second class of almost complex orbifolds we shall consider is the so-called *projective orbifold*.

**Definition 2.1.8:** A complex orbifold  $X$  is called *projective* if it can be realized as a projective variety.

## 2.2 Classification of compatible systems

In this subsection, we shall study pseudo-holomorphic maps from a Riemann surface into an almost complex orbifold in terms of  $C^\infty$  maps between orbifolds. For definitions or notations regarding good maps and compatible systems, the reader is referred to section 4.4. The main result is summarized in the following

**Proposition 2.2.1:** For any pseudo-holomorphic map  $f$  from a Riemann surface  $\Sigma$  of genus  $g$  with  $k$  marked points  $\mathbf{z} = (z_1, z_2, \dots, z_k)$  into a closed almost complex orbifold  $(X, J)$ , there are finitely many orbifold structures on  $\Sigma$  whose singular set is contained in  $\mathbf{z}$ , and for each of these orbifold structures there are finitely many pairs  $(\tilde{f}, \xi)$ , where  $\tilde{f}$  is a good map whose underlying map is  $f$ , and  $\xi$  is an isomorphism class of compatible systems of  $\tilde{f}$ . The total number is bounded from above by a constant  $C(X, g, k)$  depending only on  $X, g, k$ .

We will break the proof into several steps. First recall that a complex orbicurve is a smooth complex curve with a reduced orbifold structure. More precisely, we have the following

**Definition 2.2.2:** A complex orbicurve of genus  $g$  is a triple  $(\Sigma, \mathbf{z}, \mathbf{m})$ , where  $\Sigma$  is a smooth complex curve of genus  $g$ ,  $\mathbf{z} = (z_1, z_2, \dots, z_k)$  is a set of distinct marked points on  $\Sigma$ ,  $\mathbf{m} = (m_1, m_2, \dots, m_k)$  is a  $k$ -tuple of integers with  $m_i \geq 2$ .  $\Sigma$  is given an orbifold structure as follows: at each point  $z_i$ , a disc neighborhood of  $z_i$  is uniformized by the branched covering map  $z \rightarrow z^{m_i}$ .

We will be interested in good  $C^\infty$  maps from an orbicurve  $(\Sigma, \mathbf{z}, \mathbf{m})$  into an almost complex orbifold  $X$  which induce a pseudo-holomorphic map. We make two observations: First, since pseudo-holomorphic maps (in the classical sense) have the unique continuity property, given an orbifold structure of  $\Sigma$  with a pseudo-holomorphic map  $f$ , there is a unique  $C^\infty$  map  $\tilde{f}$  as a germ of liftings of  $f$ . Secondly, let  $\tilde{f} : (\Sigma, \mathbf{z}, \mathbf{m}) \rightarrow X$  be a good map inducing a pseudo-holomorphic map  $f$ ; for any isomorphism class of compatible systems  $\xi$ , if the group homomorphism it defines at  $z_i$ ,  $\lambda_i : \mathbf{Z}_{m_i} \rightarrow G_{f(z_i)}$ , is not monomorphic, then we can factor  $\lambda_i$  through a monomorphism  $\lambda'_i : \mathbf{Z}_{m'_i} \rightarrow G_{f(z_i)}$ , redefine the orbifold structure at  $z_i$  by  $z \rightarrow z^{m'_i}$ , and obtain a good map  $\tilde{f}' : (\Sigma, \mathbf{z}, \mathbf{m}') \rightarrow X$  with an isomorphism class of compatible systems  $\xi'$  such that the group homomorphism of  $\xi'$  at  $z_i$  is  $\lambda'_i$ , and the restriction of  $\xi'$  to  $\Sigma \setminus \{z_i\}$  is the same as that of  $\xi$ . Therefore, we will only be considering good maps with an isomorphism class of compatible systems whose group homomorphism at each point is monomorphic.

First we show that for any pseudo-holomorphic map  $f$  from a complex curve  $\Sigma$  into  $(X, J)$ , there is always an orbifold structure on  $\Sigma$  with respect to which  $f$  admits a  $C^\infty$  lifting.

**Lemma 2.2.3:** *Let  $f : (\Sigma, j) \rightarrow (X, J)$  be a non-constant pseudo-holomorphic map such that  $f(\Sigma)$  is not entirely in  $\Sigma X$ ; then there is an orbifold structure on  $\Sigma$  making it into a complex orbicurve  $(\Sigma, \mathbf{z}, \mathbf{m})$ , and  $f$  determines a unique  $C^\infty$  map  $\tilde{f} : (\Sigma, \mathbf{z}, \mathbf{m}) \rightarrow X$  as a germ of  $C^\infty$  liftings of  $f$ , which is regular, therefore is good with a unique isomorphism class of compatible systems. Moreover, the group homomorphism at each point is monomorphic.*

**Proof:** When  $f(\Sigma) \subset X_{reg}$ ,  $f$  is just a pseudo-holomorphic map in the usual sense, and the claim is trivial. Suppose  $f(\Sigma)$  is not entirely in  $\Sigma X$ . Then there are finitely many  $z_i \in \Sigma$  such that  $f(z_i) = p_i \in \Sigma X$ . We will define the orbifold structure at each  $z_i$  as follows. There is a branched covering map  $br : z \rightarrow z^{n_i}$ , a chart  $(V_{p_i}, G_{p_i}, \pi_{p_i})$  of  $X$  at  $p_i$ , and a pseudo-holomorphic map  $\tilde{f}_i : D \rightarrow V_{p_i}$  such that  $f \circ br = \pi_{p_i} \circ \tilde{f}_i$ . Let  $m_i$  be the minimum of the  $n_i$ 's. We take  $\mathbf{z} = (z_1, \dots, z_k)$  where  $z_i \in f^{-1}(\Sigma X)$  with  $m_i \neq 1$ , and  $\mathbf{m} = (m_1, \dots, m_k)$ . Then the elliptic regularity and unique continuity property of pseudo-holomorphic maps implies that  $f$  admits a  $C^\infty$  lifting  $\tilde{f} : (\Sigma, \mathbf{z}, \mathbf{m}) \rightarrow X$ , which is obviously regular. By Lemma 4.4.11,  $\tilde{f}$  is a good map with a unique isomorphism class of compatible systems. The minimality of  $m_i$  implies that the group homomorphism at each point is monomorphic.  $\square$

For the case when  $f(\Sigma) \subset \Sigma X$ , there is a set  $\{z_1, \dots, z_k\}$  such that  $f(\Sigma \setminus \{z_1, \dots, z_k\})$  is contained in a connected stratum of  $\Sigma X = \Sigma X_{gen}$ , i.e., for  $z \in \Sigma \setminus \{z_1, \dots, z_k\}$ , let  $p = f(z)$ ; there is a chart  $(V_p, G_p, \pi_p)$ , a branched covering map  $br : z \rightarrow z^m$ , a lifting  $\tilde{f}_z$  such that  $\pi_p \circ \tilde{f}_z = f \circ br$ , and  $\tilde{f}_z(D)$  lies in the fixed-point set of  $G_p$ . It is easily seen that  $br$  can be taken trivial, i.e.,  $m = 1$ . For each  $z_i$ , there is a chart at  $p_i = f(z_i)$ ,  $(V_{p_i}, G_{p_i}, \pi_{p_i})$ , a branched covering map  $br_i$ , a lifting  $\tilde{f}_i$  such that  $\pi_{p_i} \circ \tilde{f}_i = f \circ br_i$ . Now it is easily seen that given a pseudo-holomorphic map  $f : \Sigma \rightarrow X$ , there is always an orbifold structure on  $\Sigma$  such that  $f$  is lifted to a (unique)  $C^\infty$  map.

Next we will give a classification of compatible systems of a good pseudo-holomorphic map from a complex orbicurve into  $(X, J)$ , up to isomorphism.

Let  $\tilde{f} : (\Sigma, \mathbf{z}, \mathbf{m}) \rightarrow X$  be a good pseudo-holomorphic map. Let  $\xi_0$  and  $\xi$  be two isomorphism classes of compatible systems. We will fix  $\xi_0$  for a moment. The restriction to  $\Sigma \setminus \mathbf{z}$  gives rise to two isomorphism classes of compatible systems of  $\tilde{f}$  on  $\Sigma \setminus \mathbf{z}$ , which are represented by geodesic compatible systems  $\{\tilde{f}_{0,UU'}, \lambda_0\}$  and  $\{\tilde{f}_{UU'}, \lambda\}$ , respectively. In addition, we fix a base point  $z_0 \in \Sigma \setminus \mathbf{z}$  such that  $f(z_0)$  lies in the main stratum. We denote  $G_{f(z_0)}$  by  $H_{z_0}$ . We remark that since each point in  $\Sigma \setminus \mathbf{z}$  is regular, for each inclusion between elements in  $\mathcal{U}$ , there is only one injection. Now we take an element  $U_0$  in  $\mathcal{U}$  containing  $z_0$  which is maximal in the sense that  $U_0$  is not contained in any other element in  $\mathcal{U}$ . By the choice of  $z_0$ , we see that  $\tilde{f}_{0,U_0U'_0} = \tilde{f}_{U_0U'_0}$ . Each automorphism of the uniformizing system of  $U'_0$ , given by an element of  $H_{z_0}$ , leaves  $\tilde{f}_{U_0U'_0}$  fixed. We will introduce a basic operation on the compatible system  $\{\tilde{f}_{UU'}, \lambda\}$ , called *conjugation*, which will only produce an isomorphic compatible system. The operation is this: for an element  $U'$  in  $\mathcal{U}'$ , we take an automorphism  $\delta$  of the uniformizing system of  $U'$ , and change  $\tilde{f}_{UU'}$  to  $\delta \circ \tilde{f}_{UU'}$  and for each inclusion  $i : U_1 \rightarrow U$ , we change  $\lambda(i)$  to  $\delta \circ \lambda(i)$ , and for each inclusion  $i : U \rightarrow U_2$ , we change  $\lambda(i)$  to  $\lambda(i) \circ \delta^{-1}$ . For any  $U$  in  $\mathcal{U}$  such that  $U \subset U_0$ , there is a unique conjugation done to  $U$  after which we have  $\tilde{f}_{0,UU'} = \tilde{f}_{UU'}$  and  $\lambda_0(i) = \lambda(i)$  where  $i : U \rightarrow U_0$  is the inclusion. Then the compatibility condition of compatible systems (4.4.1) ensures that for any  $i : U_2 \rightarrow U_1$  where  $U_1, U_2 \subset U_0$ , we have  $\lambda_0(i) = \lambda(i)$ . In other words, we show that the two compatible systems agree on  $U_0$ . Next we will try to spread this out. We take an element  $U_1 \in \mathcal{U}$  which is maximal and has non-empty intersection with  $U_0$ , which is a geodesically convex topological disc. We pick a  $U \subset U_0 \cap U_1$ , and let  $i : U \rightarrow U_1$ ; then there is a unique conjugation done to  $U_1$  after which we have

$\tilde{f}_{0,U_1U'_1} = \tilde{f}_{U_1U'_1}$  and  $\lambda_0(i) = \lambda(i)$ . The compatibility condition (4.4.1) ensures that  $\lambda_0(j) = \lambda(j)$  for any inclusion  $j : W \rightarrow U_1$  where  $W \in \mathcal{U}$  and  $W \subset U$ , and since  $U_0 \cap U_1$  is connected, we can drop the condition  $W \subset U$ . We then proceed to do conjugations to all  $U \subset U_1$  so that the two compatible systems agree on  $U_0 \cup U_1$ . We can continue to do this as long as the new element we take in  $\mathcal{U}$  intersects the union of the old ones in a connected subset, and whenever this fails, we have a loop, and we are forced to do a conjugation to an element in  $\mathcal{U}$  that we have done before. But if the loop is homotopically trivial, no problem will be caused. Now we take a loop  $\gamma$  based at  $z_0$ , going along  $\gamma$  once; we will be forced to do a conjugation twice to an element  $U \subset U_0$  so that the equation  $\lambda_0(i) = \lambda(i)$  may not hold any longer, where  $i : U \rightarrow U_0$ . We take  $\lambda_0(i) \circ (\lambda(i))^{-1}$  which is an element in  $H_{z_0}$ , and we denote it by  $g_\gamma$ . One can verify:

- If we do a conjugation to  $U_0$  with automorphism  $\delta$ , then  $g_\gamma$  will be changed to  $\delta \circ g_\gamma \circ \delta^{-1}$ .
- If  $\gamma_1$  is homotopic to  $\gamma_2$ , then  $g_{\gamma_1} = g_{\gamma_2}$ .
- The assignment  $[\gamma] \rightarrow g_\gamma$  defines a homomorphism  $\theta_{\xi_0, \xi} : \pi_1(\Sigma \setminus \mathbf{z}, z_0) \rightarrow H_{z_0}$ .
- If  $\theta_{\xi_0, \xi_1} = \theta_{\xi_0, \xi_2}$ , then the restriction of  $\xi_1$  to  $\Sigma \setminus \mathbf{z}$  is isomorphic to that of  $\xi_2$ .

The homomorphism  $\theta_{\xi_0, \xi}$  measures the difference between  $\xi_0$  and  $\xi$  restricted to  $\Sigma \setminus \mathbf{z}$ . When  $f(\Sigma)$  lies in a uniformized open set, we can get an absolute measurement. A similar argument shows that

**Lemma 2.2.4:** *If  $f(\Sigma)$  lies in a neighborhood uniformized by  $(V, G, \pi)$ , then each isomorphism class of compatible systems determines a conjugacy class of homomorphisms  $\theta_\xi : \pi_1(\Sigma \setminus \mathbf{z}, z_0) \rightarrow G$  such that  $\theta_{\xi_1} = \theta_{\xi_2}$  implies that restricting to  $\Sigma \setminus \mathbf{z}$ ,  $\xi_1$  is isomorphic to  $\xi_2$ .*

**Proof:** Given a compatible system  $\{\tilde{f}_{UU'}, \lambda\}$ . Consider the inclusion from  $\cup_{U' \in \mathcal{U}'} U'$  into  $\pi(V)$ . By taking composition, we obtain a compatible system  $\{\tilde{f}_{WW'}, \tau\}$  which is isomorphic to  $\{\tilde{f}_{UU'}, \lambda\}$  and  $W'$  consists of a single element  $\pi(V)$  uniformized by  $(V, G, \pi)$ . Moreover, each  $\tau(i)$  is an automorphism of  $(V, G, \pi)$ , given by an element of  $G$ . Now we take a loop  $\gamma$  in  $\Sigma \setminus \mathbf{z}$  based at  $z_0$ , and cover  $\gamma$  by a set of discs in  $\mathcal{W}$ , say  $W_1, \dots, W_k$ , such that only two adjacent discs and  $W_1, W_k$  have non-empty intersection. For any  $i \in \{1, \dots, k\}$ , take a point  $z_i \in W_i \cap W_{i+1}$  (if  $i = k$ , then  $i + 1 = 1$ ) and a disc neighborhood  $D_i \in \mathcal{W}$  of it such that  $D_i \subset W_i \cap W_{i+1}$ . Let  $a_i : D_i \rightarrow W_i$ ,  $b_i : D_i \rightarrow W_{i+1}$  be the inclusions respectively; we put  $g_i = \lambda_1(b_i) \circ \lambda_1^{-1}(a_i)$  in  $G$ , and  $g_\gamma = g_k \circ \dots \circ g_1$ . Then one can verify that the assignment  $\gamma \rightarrow g_\gamma$  is well-defined and gives the required homomorphism  $\theta_\xi : \pi_1(\Sigma \setminus \mathbf{z}, z_0) \rightarrow G$ .  $\square$

**Definition 2.2.5:**  $\theta_\xi$  is called the characteristic of  $\xi$ .  $\theta_{\xi_0, \xi_1}$  is called the difference characteristic of the pair  $(\theta_0, \theta_1)$ .

The next lemma deals with compatible systems defined near an orbifold point.

**Lemma 2.2.6:** *Suppose  $\tilde{f} : D \setminus \{0\} \rightarrow U_p$  is a good pseudo-holomorphic map which can be extended continuously over  $D$  sending 0 to  $p$ , where  $U_p$  is a geodesic neighborhood with a geodesic chart  $(V_p, G_p, \pi_p)$ , and suppose each point in  $f(D \setminus \{0\})$  has the same orbit type given by the conjugacy class  $(H)$  of subgroups of  $G_p$ . For any isomorphism class of compatible systems  $\xi$  of  $\tilde{f}$ , let  $\theta_\xi : \pi_1(D \setminus \{0\}, *) = \mathbf{Z} \rightarrow G_p$  be given by  $x \rightarrow g_\xi$  where  $x$  is the positively oriented generator; then after choosing a representative  $H$ ,  $g_\xi$  lies in the normalizer  $N_{G_p}(H)$  of  $H$  in  $G_p$ . Let  $m$  be the order of  $g_\xi$  in  $G_p$ . Then we can give an orbifold structure at  $\{0\}$  to  $D$  by  $z \rightarrow z^m$ , so that  $\tilde{f}$  is extended to a good map  $\tilde{f}'$  from  $(D, (0), (m))$  to  $U_p$  with a unique isomorphism class of compatible*



systems  $\xi'$ , such that the group homomorphism of  $\xi'$  at 0 is given by  $e^{2\pi i/m} \rightarrow g_\xi$ , and we have  $\theta_\xi = \theta_{\xi'}$ .

**Proof:** Choose a representative  $H$  of  $(H)$ . From the construction of the homomorphism  $\theta_\xi : \pi_1(D \setminus \{0\}, *) \rightarrow G_p$  (cf. Lemma 2.2.4), and the assumption that each point in  $f(D \setminus \{0\})$  has the same orbit type given by the conjugacy class  $(H)$ , we see that the element  $g_\xi \in G_p$  must be in the normalizer  $N_{G_p}(H)$  of  $H$  in  $G_p$ . Consider the preimage of  $f(D \setminus \{0\})$  in  $V_p$  under  $\pi_p$ . Since each point in  $f(D \setminus \{0\})$  has the same orbit type given by the conjugacy class of the subgroup  $H$  of  $G_p$ , we consider the part  $W$  of the preimage lying in the fixed-point set of  $H$ ,  $V^H$ , in  $V_p$ . We take a non-trivial loop in  $D \setminus \{0\}$  such that  $f$  is an embedding in a neighborhood of it; then it is easily seen that one of the components of its preimage in  $W$  cyclically covers it under  $\pi_p$ , which is given by the action of  $[g_\xi]$  in  $N_{G_p}(H)/H$ . Therefore there is a lifting of  $f$ ,  $\tilde{f} : D \setminus \{0\} \rightarrow V_p$ , a branched covering map  $br : z \rightarrow z^m$  where  $m$  equals the order of  $g_\xi$ , such that  $\pi_p \circ \tilde{f} = f \circ br$ , and  $\tilde{f}$  is equivariant under the homomorphism  $e^{2\pi i/m} \rightarrow g_\xi$ . By removability of the singularity,  $\tilde{f}$  extends to a pseudo-holomorphic map  $D \rightarrow V_p$ , still denoted by  $\tilde{f}$ . Now we need to extend the compatible system  $\xi$  defined over  $D \setminus \{0\}$  to a compatible system  $\xi'$  on  $D$ , including  $(\tilde{f}, \rho)$  where  $\rho : \mathbf{Z}_m \rightarrow G_p$  is given by  $e^{2\pi i/m} \rightarrow g_\xi$ . Let  $\xi$  be given by  $\{\tilde{f}_{WW'}, \lambda\}$ . We take an element  $W \in \mathcal{W}$  and an injection  $i$  from  $W$  into  $D$ , and we define  $\lambda(i)$  to be an injection from the uniformizing system of  $W'$  into  $V_p$  such that  $\tilde{f} \circ i = \lambda(i) \circ \tilde{f}_{WW'}$ . Then we can extend out to define  $\lambda(j)$  for injections  $j : W \rightarrow D$  nearby  $i$ . One can verify that  $\lambda(i)$  can be chosen so that we can make it consistent when we move around a loop. This gives the extension  $\xi'$  of  $\xi$ . The statement that  $\xi'$  is unique and  $\theta_\xi = \theta_{\xi'}$  follows from the fact that the compatible system on  $D \setminus \{0\}$  induced by any map  $\tilde{f} : D \rightarrow V_p$  with homomorphism  $e^{2\pi i/m} \rightarrow g$  has the corresponding  $\theta : \pi_1(D \setminus \{0\}, *) \rightarrow G_p$  given by  $x \rightarrow g$ .  $\square$

With Lemma 2.2.6 understood, one can regard the characteristics  $\theta_\xi$  or  $\theta_{\xi_0, \xi_1}$  as defined on the orbifold fundamental group  $\pi_1^{orb}(\Sigma, \mathbf{z}, \mathbf{m}, z_0)$  after giving a reduced orbifold structure on  $(\Sigma, \mathbf{z})$  according to Lemma 2.2.6. In the case when  $f(\Sigma)$  lies in a global quotient  $U = V/G$ , given any compatible system  $\xi$ , let  $G_\xi$  be the image of  $\theta_\xi$  in  $G$ . Then there is a smooth Riemann surface  $\Sigma_\xi$  with an action of  $G_\xi$ , such that  $\pi_1(\Sigma_\xi)$  is the kernel of  $\theta_\xi$  and  $(\Sigma_\xi, G_\xi)$  is a uniformizing system of  $(\Sigma, \mathbf{z}, \mathbf{m})$ . Moreover, the compatible system  $\xi$  is induced from a natural equivariant map  $\tilde{f} : \Sigma_\xi \rightarrow V$  (recall that here  $G_\xi$  is a subgroup of  $G$ ).

**Corollary 2.2.7:** *Let  $f : \Sigma \rightarrow X$  be a pseudo-holomorphic map, let  $\mathbf{z}$  be a set of finitely many points in  $\Sigma$ . Then there are only finitely many orbifold structures on  $\Sigma$  having  $\mathbf{z}$  as singular set, such that  $f$  is lifted to a (unique) good map, and for each such orbifold structure there are only finitely many isomorphism classes of compatible systems. The total number of orbifold structures and isomorphism classes of compatible systems is bounded by a number depending only on  $g, k$  and  $X$ , where  $g$  is the genus of  $\Sigma$ , and  $k$  equals the order of the set  $\mathbf{z}$ .*

**Remark 2.2.8:** *In string theory, the pseudo-holomorphic map  $f : (\Sigma, \mathbf{z}, \mathbf{m}) \rightarrow X$  near a point  $z_k \in \mathbf{z}$  is regarded as open string propagation on  $V_{f(z_k)}$  with the two ends of the open string satisfying  $x(2\pi) = g \cdot x(0)$  for some element  $g \in G_{f(z_k)}$ . Likewise, closed string propagation in the quotient space  $X = Y/G$  may be regarded as open string propagation in  $Y$  with a boundary condition  $x(2\pi) = g \cdot x(0)$  for some  $1 \neq g \in G$ . These are called twisted boundary conditions in string theory. See [DHVW]. So when  $f(\Sigma)$  lies entirely in  $\Sigma X$ , it may happen that although the open string propagation is the same geometrically, it is attached with different twisted boundary conditions. On the other hand, mathematically, the different choices of twisted boundary conditions correspond to different choices of compatible system, which controls the deformation of the pseudo-holomorphic map.*

The last ingredient in the proof of Proposition 2.2.1 is provided by

**Lemma 2.2.9:** *For any pseudo-holomorphic map  $f : \Sigma \rightarrow X$ , there is an orbifold structure on  $\Sigma$  with respect to which  $f$  admits a good  $C^\infty$  lifting.*

**Proof:** We can assume the case that  $f(\Sigma)$  lies entirely in the singular set  $\Sigma X$ , and there is a set  $\{z_1, \dots, z_k\}$  such that  $f(\Sigma \setminus \{z_1, \dots, z_k\})$  is contained in a connected stratum of  $\Sigma X = \bar{\Sigma} X_{gen}$ , so that for  $z \in \Sigma \setminus \{z_1, \dots, z_k\}$ ,  $f$  can be lifted to a chart  $(V_p, G_p, \pi_p)$  where  $p = f(z)$  without using any branched covering map. For each  $z_i$ , there is a chart  $(V_{p_i}, G_{p_i}, \pi_{p_i})$  at  $p_i = f(z_i)$ , a branched covering map  $br_i$ , and a lifting  $\tilde{f}_i$  such that  $\pi_{p_i} \circ \tilde{f}_i = f \circ br_i$ . There is a subgroup  $H_i$  of  $G_{p_i}$  and an element  $a_i \in N_{G_{p_i}}(H_i)/H_i$  such that  $\tilde{f}_i$  is  $a_i$ -equivariant. We take any  $g_i \in N_{G_{p_i}}(H_i)$  such that  $[g_i] = a_i$ , and let  $m_i$  be the order of  $g_i$ . We can let  $br_i$  be given by  $z \rightarrow z^{m_i}$ . Let  $\tilde{f}_i$  be the lifting of  $f$  that is  $\lambda_i$ -equivariant, where  $\lambda_i : e^{2\pi i/m_i} \rightarrow g_i$ . On the other hand, the general construction of geodesic compatible systems can be applied here to produce a collection of maps  $\{\tilde{f}_{UU'}\}$  including these  $\tilde{f}_i$ 's, but with the assignment  $\lambda$  missing. However, the homomorphisms  $\lambda_i$  define a compatible system locally near  $z_i$ , the idea of the proof is to extend out over the whole  $\Sigma$  at the price of introducing some extra orbifold points.

Here is the basic observation: Suppose a compatible system is already built on an open subset  $\Sigma_0$  of  $\Sigma$ , and let  $U \in \mathcal{U}$  which is not contained in  $\Sigma_0$ . We want to extend the compatible system to  $\Sigma_0 \cup U$ . This consists of two steps. The first one is to assign an injection  $\lambda(i)$  between the uniformizing systems of  $U_1$  and  $U$  to any inclusion  $i : U_1 \rightarrow U$  where  $U_1 \subset \Sigma_0 \cap U$  and  $U_1 \in \mathcal{U}$ , which extends the old assignment and satisfies the compatibility condition (4.4.1). The second one is to extend over for any inclusion  $U_1 \rightarrow U$ . This can be done as long as the intersection  $U \cap \Sigma_0$  is simply connected (it could be disconnected). So after removing finitely many points, a compatible system can be constructed on  $\Sigma$ , then adding these points as orbifold points we extend the compatible system to the whole  $\Sigma$ . We remark that these extra orbifold points can be squeezed into one point, and then absorbed into one of the  $z_i$ 's.  $\square$

We end this subsection with a classification of compatible systems of a constant (pseudo-holomorphic) map. Let  $f : (\Sigma, \mathbf{z}) \rightarrow (X, J)$  be a constant map with  $f(\Sigma) = \{p\}$ . The matter is trivial if  $p \in X_{reg}$ . So we assume the case when  $p$  is in  $\Sigma X$ . The orbifold structure of  $X$  at  $p$  determines a faithful representation  $\rho : G_p \rightarrow \text{Aut}(\mathbf{C}^n)$  (here  $2n = \dim X$ ). We know that an isomorphism class of compatible systems of  $f$  is determined by a conjugacy class of homomorphisms  $\pi_1(\Sigma \setminus \mathbf{z}, *) \rightarrow G_p$ . We will next show that

**Proposition 2.2.10:** *Given a conjugacy class of homomorphisms  $\theta : \pi_1(\Sigma \setminus \mathbf{z}, *) \rightarrow G_p$ , there is a unique orbifold structure on  $\Sigma$  and a (unique) good map  $\tilde{f}$  with a unique isomorphism class of compatible systems determined by its characteristic  $\theta$ .*

**Proof:** We need a digression first, recalling from [CR1]. Let  $(\Sigma, \mathbf{z}, \mathbf{m})$  be a closed 2-dimensional orbifold, where  $\mathbf{z} = (z_1, \dots, z_k)$  and  $\mathbf{m} = (m_1, \dots, m_k)$ . Let  $E$  be a complex orbifold bundle of rank  $n$  over  $(\Sigma, \mathbf{z}, \mathbf{m})$ . Then at each singular point  $z_i$ ,  $i = 1, \dots, k$ ,  $E$  determines a representation  $\rho_i : \mathbf{Z}_{m_i} \rightarrow \text{Aut}(\mathbf{C}^n)$  so that over a disc neighborhood of  $z_i$ ,  $E$  is uniformized by  $(D \times \mathbf{C}^n, \mathbf{Z}_{m_i}, \pi)$  where the action of  $\mathbf{Z}_{m_i}$  on  $D \times \mathbf{C}^n$  is given by

$$e^{2\pi i/m_i} \cdot (z, w) = (e^{2\pi i/m_i} z, \rho_i(e^{2\pi i/m_i})w)$$

for any  $w \in \mathbf{C}^n$ . Each representation  $\rho_i$  is uniquely determined by  $n$ -tuple of integers  $(m_{i,1}, \dots, m_{i,n})$  with  $0 \leq m_{i,j} < m_i$ , as it is given by the matrix

$$\rho_i(e^{2\pi i/m_i}) = \text{diag}(e^{2\pi i m_{i,1}/m_i}, \dots, e^{2\pi i m_{i,n}/m_i}).$$

Over the punctured disc  $D_i \setminus \{0\}$  at  $z_i$ ,  $E$  inherits a specific trivialization from  $(D \times \mathbf{C}^n, \mathbf{Z}_{m_i}, \pi)$  as

follows: We define a  $\mathbf{Z}_{m_i}$ -equivariant map  $\Psi_i : D \setminus \{0\} \times \mathbf{C}^n \rightarrow D \setminus \{0\} \times \mathbf{C}^n$  by

$$(z, w_1, \dots, w_n) \rightarrow (z^{m_i}, z^{-m_{i,1}} w_1, \dots, z^{-m_{i,n}} w_n),$$

where  $\mathbf{Z}_{m_i}$  acts trivially on the second  $D \setminus \{0\} \times \mathbf{C}^n$ . Hence  $\Psi_i$  induces a trivialization  $\psi_i : E_{D_i \setminus \{0\}} \rightarrow D_i \setminus \{0\} \times \mathbf{C}^n$ . We can extend the smooth complex vector bundle  $E_{\Sigma \setminus \mathbf{z}}$  over  $\Sigma \setminus \mathbf{z}$  to a smooth complex vector bundle over  $\Sigma$  by using these trivializations  $\psi_i$ . We call the resulting complex vector bundle the *de-singularization* of  $E$ , and denote it by  $|E|$ . End of digression.

We denote by  $T_p$  the tangent space of  $V_p$  at  $p$ , and by  $\Sigma'$  the universal cover of  $\Sigma \setminus \mathbf{z}$ . We let  $E$  be the complex vector bundle over  $\Sigma \setminus \mathbf{z}$  obtained from  $\Sigma' \times T_p$  through  $\theta : \pi_1(\Sigma \setminus \mathbf{z}, *) \rightarrow G_p$  and the action of  $G_p$  on  $T_p$ . Then  $E$  naturally extends to an orbifold bundle over  $(\Sigma, \mathbf{z}, \mathbf{m})$ , still denoted by  $E$ , where  $\mathbf{m} = (m_i)$  and  $m_i$  equals the order of  $\theta(x_i)$  in  $G_p$  for the positively oriented generator  $x_i$  around  $z_i \in \mathbf{z}$ . Take a  $C^\infty$  section  $\tilde{s}$  of  $E$  as follows. Over each uniformizing system  $(D \times \mathbf{C}^n, \mathbf{Z}_{m_i}, \pi)$  of  $E$  at  $z_i \in \mathbf{z}$ ,  $\tilde{s}$  is given by  $\tilde{s}_i(z) = (\epsilon_{i,1} z^{m_{i,1}}, \dots, \epsilon_{i,n} z^{m_{i,n}})$ . We choose a generic non-zero  $(\epsilon_{i,1}, \dots, \epsilon_{i,n})$  such that  $\tilde{s}_i$  meets the fixed-point set of  $G_p$  at isolated points. Through the trivialization  $\psi_i$ , each  $\tilde{s}_i$  is identified with the constant section  $(\epsilon_{i,1}, \dots, \epsilon_{i,n})$  in the trivialization of  $|E|$  at  $z_i$ . So we let  $\tilde{s}$  equal a smooth section of  $|E|$  extending the constant section  $(\epsilon_{i,1}, \dots, \epsilon_{i,n})$  near each  $z_i$  such that over only finitely many points  $\tilde{s}$  lies in the fixed-point set of  $G_p$  in the tangent space of  $V_p$  at  $p$ . This can be done since the fixed-point set is of codimension at least two. Now we consider a  $C^\infty$  map  $\tilde{F} : (\Sigma, \mathbf{z}, \mathbf{m}) \times [0, 1] \rightarrow U_p$  given by  $(z, t) \rightarrow \text{Exp}_p \circ (t \cdot \tilde{s}(z))$ , where  $\text{Exp}_p : TX_p \rightarrow U_p$  is the exponential map at  $p$ . From the construction of  $\tilde{s}$ , we know that  $\tilde{F}$  is a good map with a unique isomorphism class of compatible systems, which induces a compatible system of  $f$ . One can check that it is the one determined by  $\theta$ .  $\square$

**Remark 2.2.11:** *In what follows we will consider pseudo-holomorphic maps from a marked Riemann surface  $(\Sigma, \mathbf{z})$  (or more general a marked nodal Riemann surface) into an almost complex orbifold  $(X, J)$ . We introduce the following convention: We see that in Definition 2.1.3 there are only finitely many points in  $\Sigma$  for which the multiplicity  $m$  is greater than one. It is our convention that these point are in  $\mathbf{z}$  (i.e. marked). On the other hand, we will call an orbifold structure on  $\Sigma$  with the set of orbifold points contained in  $\mathbf{z}$  together with an isomorphism class of compatible systems of the  $C^\infty$  lifting  $\tilde{f}$  of  $f$  with respect to this orbifold structure a twisted boundary condition of  $f$ , borrowing terminology from physics. Then Proposition 2.2.1 can be rephrased as saying that the set of twisted boundary conditions of a pseudo-holomorphic map from a marked Riemann surface of genus  $g$  with  $k$  marked points into a closed almost complex orbifold  $X$  is non-empty and its cardinality is bounded by a number  $C(X, g, k)$  depending only on  $X, g, k$ .  $\square$*

## 2.3 The moduli space of orbifold stable maps

We first recall

**Definition 2.3.1:** *A nodal curve with  $k$  marked points is a pair  $(\Sigma, \mathbf{z})$  of a connected topological space  $\Sigma = \bigcup \pi_{\Sigma_\nu}(\Sigma_\nu)$  where  $\Sigma_\nu$  is a smooth complex curve and  $\pi_\nu : \Sigma_\nu \rightarrow \Sigma$  is a continuous map, and  $\mathbf{z} = (z_1, \dots, z_k)$  are  $k$ -distinct points in  $\Sigma$  with the following properties.*

- *For each  $z \in \Sigma_\nu$ , there is a neighborhood of it such that the restriction of  $\pi_\nu : \Sigma_\nu \rightarrow \Sigma$  to this neighborhood is a homeomorphism to its image.*
- *For each  $z \in \Sigma$ , we have  $\sum_\nu \# \pi_\nu^{-1}(z) \leq 2$ .*
- *$\sum_\nu \# \pi_\nu^{-1}(z_i) = 1$  for each  $z_i \in \mathbf{z}$ .*

- The number of complex curves  $\Sigma_\nu$  is finite.
- The set of nodal points  $\{z \mid \sum_\nu \#\pi_\nu^{-1}(z) = 2\}$  is finite.

A point  $z \in \Sigma_\nu$  is called *singular* if  $\sum_\omega \#\pi_\omega^{-1}(\pi_\nu(z)) = 2$ . A point  $z \in \Sigma_\nu$  is said to be a *marked point* if  $\pi_\nu(z) = z_i \in \mathbf{z}$ . Each  $\Sigma_\nu$  is called a *component* of  $\Sigma$ . Let  $k_\nu$  be the number of points on  $\Sigma_\nu$  which are either singular or marked, and  $g_\nu$  be the genus of  $\Sigma_\nu$ ; a nodal curve  $(\Sigma, \mathbf{z})$  is called *stable* if  $k_\nu + 2g_\nu \geq 3$  holds for each component  $\Sigma_\nu$  of  $\Sigma$ .

A map  $\vartheta : \Sigma \rightarrow \Sigma'$  between two nodal curves is called an *isomorphism* if it is a homeomorphism and if it can be lifted to biholomorphisms  $\vartheta_{\nu\omega} : \Sigma_\nu \rightarrow \Sigma'_\omega$  for each component  $\Sigma_\nu$  of  $\Sigma$ . If  $\Sigma, \Sigma'$  have marked points  $\mathbf{z} = (z_1, \dots, z_k)$  and  $\mathbf{z}' = (z'_1, \dots, z'_k)$  then we require  $\vartheta(z_i) = z'_i$  for each  $i$ . Let  $Aut(\Sigma, \mathbf{z})$  be the group of automorphisms of  $(\Sigma, \mathbf{z})$ .

Each nodal curve  $(\Sigma, \mathbf{z})$  is canonically associated with a graph  $T_\Sigma$  as follows. The vertices of  $T_\Sigma$  correspond to the components of  $\Sigma$  and for each pair of components intersecting each other in  $\Sigma$  there is an edge joining the corresponding two vertices. For each point  $z \in \Sigma$  such that  $\#\pi_\nu^{-1}(z) = 2$ , there is an edge joining the same vertex corresponding to  $\Sigma_\nu$ . For each marked point, there is a half open edge (tail) attaching to the vertex. The graph  $T_\Sigma$  is connected since  $\Sigma$  is connected. We can smooth out all the nodal points to obtain a smooth surface. Its genus is called arithmetic genus of  $\Sigma$ . The arithmetic genus can be computed by the formula

$$g = \sum_\nu g_\nu + rank H_1(T; \mathbf{Q}).$$

For any real numbers  $t \geq 0, r > 0$ , set  $X(t, r) = \{(x, y) \in \mathbf{C}^2 \mid \|x\|, \|y\| < r, xy = t\}$ . We fix an action of  $\mathbf{Z}_m$  on  $X(t, r)$  for any  $m \geq 1$  by  $e^{2\pi i/m} \cdot (x, y) = (e^{2\pi i/m}x, e^{-2\pi i/m}y)$ . Observe that the (branched) covering map  $X(t, r) \rightarrow X(t^m, r^m)$  given by  $(x, y) \rightarrow (x^m, y^m)$  is  $\mathbf{Z}_m$ -invariant, hence  $(X(t, r), \mathbf{Z}_m)$  can be regarded as a “uniformizing system” of  $X(t^m, r^m)$ .

**Definition 2.3.2:** A *nodal orbicurve* is a nodal marked curve  $(\Sigma, \mathbf{z})$  with an orbifold structure as follows:

- The singular set  $\mathbf{z}_\nu = \Sigma \cap \Sigma_\nu$  (in the sense of orbifolds) of each component  $\Sigma_\nu$  is contained in the set of marked points and nodal points  $\mathbf{z}$ .
- A disc neighborhood of a marked point is uniformized by a branched covering map  $z \rightarrow z^{m_i}$ .
- A neighborhood of a nodal point is uniformized by  $(X(0, r_j), \mathbf{Z}_{n_j})$ .

Here  $m_i$  and  $n_j$  are allowed to be equal to one, i.e., the corresponding orbifold structure is trivial there. We denote the corresponding nodal orbicurve by  $(\Sigma, \mathbf{z}, \mathbf{m}, \mathbf{n})$  where  $\mathbf{m} = (m_1, \dots, m_k)$  and  $\mathbf{n} = (n_j)$ .

An *isomorphism* between two nodal orbicurves  $\tilde{\vartheta} : (\Sigma, \mathbf{z}, \mathbf{m}, \mathbf{n}) \rightarrow (\Sigma', \mathbf{z}', \mathbf{m}', \mathbf{n}')$  is a collection of  $C^\infty$  isomorphisms  $\tilde{\vartheta}_{\nu\omega}$  between the orbicurves  $\Sigma_\nu$  and  $\Sigma'_\omega$  which induces an isomorphism  $\vartheta : (\Sigma, \mathbf{z}) \rightarrow (\Sigma', \mathbf{z}')$ . The *group of automorphisms* of a nodal orbicurve  $(\Sigma, \mathbf{z}, \mathbf{m}, \mathbf{n})$  is denoted by  $Aut(\Sigma, \mathbf{z}, \mathbf{m}, \mathbf{n})$ . It is easily seen that  $Aut(\Sigma, \mathbf{z}, \mathbf{m}, \mathbf{n})$  is a subgroup of  $Aut(\Sigma, \mathbf{z})$  of finite index.

**Definition 2.3.3:** Let  $(X, J)$  be an almost complex orbifold. An *orbifold stable map* is a triple  $(f, (\Sigma, \mathbf{z}, \mathbf{m}, \mathbf{n}), \xi)$  described as follows:

1.  $f$  is a continuous map from a nodal orbicurve  $(\Sigma, \mathbf{z})$  into  $X$  such that each  $f_\nu = f \circ \pi_\nu$  is a pseudo-holomorphic map from  $\Sigma_\nu$  into  $X$ .

2.  $\xi$  is an isomorphism class of compatible structures defined in the same way as in the smooth case.
3. At both marked and nodal points, the induced homomorphism on the local group is injective.
4. Let  $k_\nu$  be the order of the set  $\mathbf{z}_\nu$ , namely the number of points on  $\Sigma_\nu$  which are singular (i.e. nodal or marked ); if  $f_\nu$  is a constant map, then  $k_\nu + 2g_\nu \geq 3$ .

(We will call  $\xi$  a twisted boundary condition of  $f : (\Sigma, \mathbf{z}) \rightarrow X$ ; cf. Remark 2.2.11.)

We introduce an equivalence relation amongst the set of stable maps as follows: two stable maps  $(f, (\Sigma, \mathbf{z}), \xi)$  and  $(f', (\Sigma', \mathbf{z}'), \xi')$  are *equivalent* if there exists an isomorphism  $\vartheta : (\Sigma, \mathbf{z}, \mathbf{m}, \mathbf{n}) \rightarrow (\Sigma', \mathbf{z}', \mathbf{m}', \mathbf{n}')$  such that  $f' \circ \vartheta = f$ , and the compatible systems defined by  $\xi'$  pull back via  $\vartheta$  to compatible systems isomorphic to the ones defined by  $\xi$  (we write this as  $\xi' \circ \vartheta = \xi$ ). The *automorphism group* of a stable map  $(f, (\Sigma, \mathbf{z}, \mathbf{n}), \xi)$ , denoted by  $\text{Aut}(f, (\Sigma, \mathbf{z}), \xi)$ , is defined by

$$\text{Aut}(f, (\Sigma, \mathbf{z}), \xi) = \{\vartheta \in \text{Aut}(\Sigma, \mathbf{z}, \mathbf{m}, \mathbf{n}) \mid f \circ \vartheta = f, \xi \circ \vartheta = \xi\}.$$

We often drop  $\mathbf{m}, \mathbf{n}$  to simplify the notation.

The proof of the following lemma is routine and is left to the reader.

**Lemma 2.3.4:** *The automorphism group of an orbifold stable map is finite.*

In the smooth case, the automorphism group is the only source of orbifold structure of the moduli space of stable maps. In the orbifold case, orbifold structure introduces an additional orbifold structure to the moduli space of orbifold stable maps (see formula (3.2.1)).

Given a stable map  $(f, (\Sigma, \mathbf{z}), \xi)$ , there is an associated homology class  $f_*([\Sigma])$  in  $H_2(X; \mathbf{Z})$  defined by  $f_*([\Sigma]) = \sum_\nu (f \circ \pi_\nu)_*[\Sigma_\nu]$ . On the other hand, for each marked point  $z$  on  $\Sigma_\nu$ , say  $\pi_\nu(z) = z_i \in \mathbf{z}$ ,  $\xi_\nu$  determines, by the group homomorphism at  $z$ , a conjugacy class  $(g_i)$ , where  $g_i \in G_{f(z_i)}$ . We thus have a map  $ev$  sending each (equivalence class of) stable map into  $\tilde{X}^k$  by  $(f, (\Sigma, \mathbf{z}), \xi) \rightarrow ((f(z_1), (g_1)), \dots, (f(z_k), (g_k)))$ . Here  $\tilde{X} = \{(p, (g)_{G_p}) \mid p \in X, g \in G_p\}$ , cf. [CR1]. Its components are called twisted sectors and are indexed by the equivalence class of the conjugacy class  $(g)_{G_p}$  under injection (See [CR1] for details. Let  $\mathbf{x} = \prod_i X_{(g_i)}$  be a connected component in  $\tilde{X}^k$ .

**Definition 2.3.5:** *A stable map  $(f, (\Sigma, \mathbf{z}), \xi)$  is said to be of type  $\mathbf{x}$  if  $ev((f, (\Sigma, \mathbf{z}), \xi)) \in \mathbf{x}$ . Given a homology class  $A \in H_2(X; \mathbf{Z})$ , we let  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  denote the moduli space of equivalence classes of orbifold stable maps of genus  $g$ , with  $k$  marked points, and of homology class  $A$  and type  $\mathbf{x}$ , i.e.,*

$$\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x}) = \{[(f, (\Sigma, \mathbf{z}), \xi)] \mid g_\Sigma = g, \#\mathbf{z} = k, f_*([\Sigma]) = A, ev((f, (\Sigma, \mathbf{z}), \xi)) \in \mathbf{x}\}.$$

The rest of this subsection is devoted to giving a topology on  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  and to proving that the moduli space is compact when  $(X, J)$  is a compact symplectic orbifold or a projective orbifold.

The set of all isomorphism classes of stable curves of genus  $g$  with  $k$  marked points, denoted by  $\overline{\mathcal{M}}_{g,k}$ , is called the *Deligne-Mumford compactification* of the moduli space  $\mathcal{M}_{g,k}$  of Riemann surfaces of genus  $g$  with  $k$  marked points (assuming  $k+2g \geq 3$ ). The following differential geometric description of  $\overline{\mathcal{M}}_{g,k}$  is standard.

The moduli space  $\overline{\mathcal{M}}_{g,k}$  admits a stratification which is indexed by the combinatorial types of the stable curves. More precisely, we consider a connected graph  $T$  together with a non-negative

integer  $g_\nu$ . Let  $k_\nu$  be the number of edges containing  $\nu$  (we count twice the edges both of whose vertices are  $\nu$ ). Then the data is required to satisfy

$$k_\nu + 2g_\nu \geq 3, \text{ and } \sum_\nu g_\nu + \text{rank } H_1(T; \mathbf{Q}) = g.$$

Let  $\text{Comb}(g, k)$  be the set of all such objects  $(T, (g_\nu))$ . For each element  $(\Sigma, \mathbf{z}) \in \overline{\mathcal{M}}_{g,k}$ , there is an associated element of  $\text{Comb}(g, k)$  as follows: we take the graph  $T = T_\Sigma$ , and let  $g_\nu$  be the genus of  $\Sigma_\nu$ . The set of combinatorial types  $\text{Comb}(g, k)$  is known to be of finite order ([FO]).

There is a partial order  $\succ$  on  $\text{Comb}(g, k)$  defined as follows. Let  $(T, (g_\nu)) \in \text{Comb}(g, k)$ . We consider  $(T_\nu, (g_{\nu\omega})) \in \text{Comb}(g_\nu, k_\nu)$  for some of the vertices  $\nu = \nu_1, \dots, \nu_a$  of  $T$ . We replace the vertex  $\nu$  of  $T$  by the graph  $T_\nu$ , and join the edge containing  $\nu$  to the vertex  $o_\nu(i)$  where  $i \in \{1, \dots, k_\nu\}$  is the suffix corresponding to this edge. We then obtain a new graph  $\tilde{T}$ . The number  $\tilde{g}_\nu$  is determined from  $g_\nu$  and  $g_{\nu\omega}$  in an obvious way. It is easily seen that  $(\tilde{T}, (\tilde{g}_\nu), \tilde{o})$  is in  $\text{Comb}(g, k)$ . We then define  $(T, (g_\nu), o) \succ (\tilde{T}, (\tilde{g}_\nu), \tilde{o})$ .

The structure of the Deligne-Mumford compactification  $\overline{\mathcal{M}}_{g,k}$  is summarized in the following

**Proposition 2.3.6:** ([FO]) *Let  $\mathcal{M}_{g,k}(T, (g_\nu))$  be the set of all stable curves such that the associated object is  $(T, (g_\nu))$ . Then*

- $\overline{\mathcal{M}}_{g,k}$  is a compact complex orbifold which admits a stratification with finitely many strata, and each stratum is of the form  $\mathcal{M}_{g,k}(T, (g_\nu))$ .
- There is a fiber bundle  $\mathcal{U}_{g,k}(T, (g_\nu)) \rightarrow \mathcal{M}_{g,k}(T, (g_\nu))$  which has the following property. For each  $x = (\Sigma_x, \mathbf{z}_x) \in \mathcal{M}_{g,k}(T, (g_\nu))$ , there is a neighborhood of  $x$  in  $\mathcal{M}_{g,k}(T, (g_\nu))$  of the form  $U_x = V_x/G_x$ , where  $G_x = \text{Aut}(\Sigma_x, \mathbf{z}_x)$ , such that the inverse image of  $U_x$  in  $\mathcal{U}_{g,k}(T, (g_\nu))$  is diffeomorphic to  $V_x \times \Sigma_x/G_x$ . There is a complex structure on each fiber such that the fiber of  $y = (\Sigma_y, \mathbf{z}_y)$  is identified with  $(\Sigma_y, \mathbf{z}_y)$  itself, together with a Kähler metric  $\mu_y$  which is flat in a neighborhood of the singular points and varies smoothly in  $y$ .
- $\mathcal{M}_{g,k}(T', (g'_\nu))$  is contained in the compactification of  $\mathcal{M}_{g,k}(T, (g_\nu))$  in  $\overline{\mathcal{M}}_{g,k}$  only if  $(T, (g_\nu)) \succ (T', (g'_\nu))$ .
- Different strata are patched together in a way which is described in the following local model of a neighborhood of a stable curve in  $\overline{\mathcal{M}}_{g,k}$ . A neighborhood of  $x = (\Sigma, \mathbf{z})$  in  $\overline{\mathcal{M}}_{g,k}$  is parametrized by

$$\frac{V_x \times B_r(\oplus_z T_{z_\nu} \Sigma_\nu \otimes T_{z_\omega} \Sigma_\omega)}{\text{Aut}(\Sigma, \mathbf{z})},$$

where  $z = \pi_\nu(z_\nu) = \pi_\omega(z_\omega)$  (Here it may happen that  $\nu = \omega$ ) runs over all singular points of  $\Sigma$ , and  $B_r(W)$  denotes the ball of radius  $r$  of the vector space  $W$ . Each  $y \in V_x$  represents a stable curve  $(\Sigma_y, \mathbf{z}_y)$  homeomorphic to  $(\Sigma, \mathbf{z})$ , with a Kähler metric  $\mu_y$  which is flat in a neighborhood of the singular points. Given  $y \in V_x$ , for each element  $\varsigma = (\sigma_z) \in \oplus_z T_{z_\nu} \Sigma_\nu \otimes T_{z_\omega} \Sigma_\omega$  there is an associated stable curve  $(\Sigma_{y,\varsigma}, \mathbf{z}_{y,\varsigma})$  obtained as follows. Each component  $\Sigma_\nu$  of  $\Sigma_y$  is given a Kähler metric  $\mu_y$  which is flat in a neighborhood of the singular points. This gives a Hermitian metric on each  $T_{z_\nu} \Sigma_\nu$ . For each non-zero  $\sigma_z \in T_{z_\nu} \Sigma_\nu \otimes T_{z_\omega} \Sigma_\omega$ , there is a biholomorphic map  $\Psi_{\sigma_z} : T_{z_\nu} \Sigma_\nu \setminus \{0\} \rightarrow T_{z_\omega} \Sigma_\omega \setminus \{0\}$  defined by  $u \otimes \Psi_{\sigma_z}(u) = \sigma_z$ . Let  $|\sigma_z| = R^{-2}$ ; then for sufficiently large  $R$ , the map  $\exp_{z_\omega}^{-1} \circ \Psi_{\sigma_z} \circ \exp_{z_\nu}$  is a biholomorphism between  $D_{z_\nu}(R^{-1/2}) \setminus D_{z_\nu}(R^{-3/2})$  and  $D_{z_\omega}(R^{-1/2}) \setminus D_{z_\omega}(R^{-3/2})$ , where  $D_{z_\nu}(R^{-1/2})$  is a disc neighborhood of  $z_\nu$  in  $\Sigma_\nu$  of radius  $(R^{-1/2})$  which is flat assuming  $R$  is sufficiently large. We glue  $\Sigma_\nu$  and  $\Sigma_\omega$  by this biholomorphism. If  $\sigma_z = 0$ , we do not make any change. Thus we obtain  $(\Sigma_{y,\varsigma}, \mathbf{z}_{y,\varsigma})$ .

Moreover, there is a Kähler metric  $\mu_{y,\varsigma}$  on  $\Sigma_{y,\varsigma}$  which coincides with the Kähler metric  $\mu_y$  on  $\Sigma_y$  outside a neighborhood of the singular points, and varies smoothly in  $\varsigma$ . Each  $\gamma \in \text{Aut}(\Sigma, \mathbf{z})$  takes  $(\Sigma_y, \mathbf{z}_y)$  to  $(\Sigma_{\gamma(y)}, \mathbf{z}_{\gamma(y)})$  isometrically, so it acts on  $\oplus_z T_{z_\nu} \Sigma_\nu \otimes T_{z_\omega} \Sigma_\omega$ .  $\gamma$  induces an isomorphism between  $(\Sigma_{y,\varsigma}, \mathbf{z}_{y,\varsigma})$  and  $(\Sigma_{\gamma(y,\varsigma)}, \mathbf{z}_{\gamma(y,\varsigma)})$ , which is also an isometry.

Now we define a topology on the moduli space  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ . We put a Hermitian metric  $h$  on  $(X, J)$  and the distance function on  $X$  is assumed to be induced from  $h$ .

**Definition 2.3.7:** A sequence of equivalence classes of stable maps  $x_n$  in  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  is said to converge to  $x_0 \in \overline{\mathcal{M}}_{g,k}(X, J, A, E)$  if there are representatives  $(f_n, (\Sigma_n, \mathbf{z}_n), \xi_n)$  of  $x_n$  and a representative  $(f_0, (\Sigma_0, \mathbf{z}_0), \xi_0)$  of  $x_0$  for which the following conditions hold.

- For each  $n$  (including  $n = 0$ ), there is a set of distinct regular points  $\{z_{n,1}, \dots, z_{n,a}\}$  (it may happen that this set is empty) on  $\Sigma_n$  which is disjoint from the marked point set  $\mathbf{z}_n$  such that after adding this set to  $\mathbf{z}_n$ , we obtain a stable curve in  $\overline{\mathcal{M}}_{g,k+a}$ , denoted by  $(\Sigma_n, \mathbf{z}_n)^+$ . Let  $(f_n^+, (\Sigma_n, \mathbf{z}_n)^+, \xi_n^+)$  be the sequence of stable maps naturally obtained.
- The sequence  $(\Sigma_n, \mathbf{z}_n)^+$  converges to  $(\Sigma_0, \mathbf{z}_0)^+$  in  $\overline{\mathcal{M}}_{g,k+a}$ . This means that for sufficiently large  $n$ ,  $(\Sigma_n, \mathbf{z}_n)^+$  is identified with  $(\Sigma_{y_n, \varsigma_n}, \mathbf{z}_{y_n, \varsigma_n})$  for some  $(y_n, \varsigma_n)$  in the canonical model of a neighborhood of  $(\Sigma_0, \mathbf{z}_0)^+$ . Let  $\varsigma_n$  be given by  $(\sigma_{z,n})$  and  $|\sigma_{z,n}| = R_{z,n}^{-2}$  (here  $R_{z,n}$  is allowed to be  $\infty$ ). For each  $\mu > \max_z (R_{z,n}^{-1})$  we put

$$W_{z,n}(\mu) = (D_{z_\nu}(\mu) \setminus D_{z_\nu}(R_{z,n}^{-1})) \cup (D_{z_\omega}(\mu) \setminus D_{z_\omega}(R_{z,n}^{-1})), \text{ and } W_n(\mu) = \cup_z W_{z,n}(\mu).$$

Then the following holds. First, for each  $\mu > 0$ , when  $n$  is sufficiently large the restriction of  $\tilde{f}_n^+$  to  $\Sigma_{y_n, \varsigma_n} \setminus W_n(\mu)$  converges to  $\tilde{f}_0^+$  in the  $C^\infty$  topology as a  $C^\infty$  map with an isomorphism class of compatible systems. Secondly,  $\lim_{\mu \rightarrow 0} \limsup_{n \rightarrow \infty} \text{Diam}(f_n(W_{z,n}(\mu))) = 0$  for each singular point  $z$  of  $\Sigma_0$ .

- Suppose that  $z_0 \in \Sigma_0$  is a nodal point near where  $(f_0, \xi_0)$  is given by an equivariant map  $(\tilde{f}_0, \lambda_0) : (X(0, r_0), \mathbf{Z}_m) \rightarrow (V_p, G_p)$ , where  $p = f(z_0) \in X$  is an orbifold point with a uniformizing system  $(V_p, G_p, \pi_p)$ . Then there exist  $t_n \mapsto 0$  such that  $(f_n, \xi_n)$  can be represented locally by equivariant maps  $(\tilde{f}_n, \lambda_0) : (X(t_n, r_0), \mathbf{Z}_m) \rightarrow (V_p, G_p)$ .

**Proposition 2.3.8:** Suppose  $X$  is either a symplectic orbifold with a symplectic form  $\omega$  and an  $\omega$ -compatible almost complex structure  $J$ , or a projective orbifold with an integrable almost complex structure  $J$ . Then the moduli space  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  is compact and metrizable.

The proof follows the same lines of the proof for regular stable maps. We will focus on the difference and the rest is referred to [FO]. We first consider the case when  $X$  is a compact symplectic orbifold, with a symplectic form  $\omega$  and an  $\omega$ -compatible almost complex structure  $J$ . Then  $h_\omega(u, v) = \frac{1}{2}(\omega(u, Jv) + \omega(v, Ju))$  defines a Hermitian metric on  $(X, J)$ . Suppose  $(f, (\Sigma, \mathbf{z}), \xi)$  is a stable map. The area of  $f(\Sigma)$  in  $(X, J, h_\omega)$  is given by

$$\text{Area}_{h_\omega}(f(\Sigma)) = \sum_\nu \int_{\Sigma_\nu} \tilde{f}_\nu^* \omega.$$

**Lemma 2.3.9:** Let  $(f, (\Sigma, \mathbf{z}), \xi)$  be a stable map in  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ . The area of  $f(\Sigma)$ ,  $\text{Area}_{h_\omega}(f(\Sigma))$ , is equal to a constant depending only on the homology class  $A$  and the symplectic form  $\omega$ .

**Proof:** Let  $\tilde{f} : (\Sigma, \mathbf{z}, \mathbf{m}) \rightarrow X$  be any good  $C^\infty$  map from a complex orbicurve into  $X$  which induces a pseudo-holomorphic map  $f$ . Then using the pull-back of the tangent bundle  $TX$  and the exponential map  $Exp$ , we obtain a deformation family of  $C^\infty$  maps  $\tilde{f}_t$  for  $0 \leq t \leq \epsilon$  such that for each  $0 < t \leq \epsilon$ ,  $f_t^{-1}(\Sigma X)$  is a set of finitely many points, and  $\tilde{f}_0 = \tilde{f}$ . Moreover,  $\lim_{t \rightarrow 0} \int_\Sigma \tilde{f}_t^* \omega = \int_\Sigma \tilde{f}^* \omega$ . Each  $(f_t)_*([\Sigma])$  defines a homology class in the 2nd intersection homology of  $X$  with top perversity,  $IH_2^{\bar{t}}(X)$ . On the other hand, the restriction of  $\omega$  to  $X_{reg}$ ,  $\omega_{reg}$ , defines a functional  $\langle \cdot, [\omega] \rangle$  on  $IH_2^{\bar{t}}(X)$ , which depends only on the cohomology class of  $\omega$ , and  $\int_\Sigma \tilde{f}_t^* \omega = \langle (f_t)_*([\Sigma]), [\omega] \rangle$ . Since as an oriented orbifold  $X$  is normal, i.e.,  $H_{2n}(X, X \setminus \{p\}) = \mathbf{Z}$  (here  $2n = \dim X$ ) for any  $p \in X$ , we have  $IH_i^{\bar{t}}(X) = H_i(X)$ . So  $\langle \cdot, [\omega] \rangle$  is a functional on  $H_2(X)$ , and  $\int_\Sigma \tilde{f}^* \omega = \langle f_*([\Sigma]), [\omega] \rangle$  holds there. The lemma follows easily.  $\square$

In light of Lemma 2.3.9, we have

**Corollary 2.3.10:** *There exists a constant  $\delta > 0$  such that for any non-constant pseudo-holomorphic map  $f : \Sigma \rightarrow X$ , the  $h_\omega$ -area of  $f(\Sigma)$  is greater than  $\delta$ . Here  $\Sigma$  is a closed Riemann surface.*

The following lemma is an extended version of Lemma 11.2 in [FO] which was used in the proof of the compactness of moduli spaces of stable maps.

**Lemma 2.3.11:** *(cf. Lemma 11.2 in [FO]) There exists  $\epsilon$  independent of  $L$  and depending only on  $(X, \omega, J)$  with the following properties. If  $f : [-L, L] \times S^1 \rightarrow X$  is a pseudo-holomorphic map which admits a  $C^\infty$  lifting such that there are no orbifold points on  $[-L, L] \times S^1$ , and if  $\text{Diam}(f([-L, L] \times S^1)) < \epsilon$ , then*

$$\left| \frac{\partial f}{\partial \tau}(\tau, t) \right| + \left| \frac{\partial f}{\partial t}(\tau, t) \right| \leq C e^{-\frac{1}{m} \text{dist}(\tau, \partial[-L, L])} \quad \forall (\tau, t) \in [-L+1, L-1] \times S^1.$$

Here  $C$  and  $m$  are independent of  $L$ .

**Proof:** When the diameter of  $f([-L, L] \times S^1)$  is small enough, it lies in a uniformized open subset  $U$  of  $X$ , uniformized by  $(V, G, \pi)$ . Then there is a finite covering map  $co : [-L/m, L/m] \times S^1 \rightarrow [-L, L] \times S^1$  and a lifting  $\tilde{f} : [-L/m, L/m] \times S^1 \rightarrow V$  such that  $\pi \circ \tilde{f} = f \circ co$ . We can apply Lemma 11.2 in [FO] to  $\tilde{f}$  to get the estimate. Note that the order  $m$  of the covering map  $co$  is bounded from above by a constant depending only on  $X$ .  $\square$

Given a sequence of stable maps  $(f_n, (\Sigma_n, \mathbf{z}_n), \xi_n)$ , by adding finitely many additional marked points (whose number is independent of  $n$ ), we may assume that  $(\Sigma_n, \mathbf{z}_n)$  are stable curves. Then a subsequence of  $(\Sigma_n, \mathbf{z}_n)$  converges in the Deligne-Mumford compactification, so that there is a  $(\Sigma_0, \mathbf{z}_0)$ , for large  $n$ , such that  $(\Sigma_n, \mathbf{z}_n)$  is identified with  $(\Sigma_{y_n, \varsigma_n}, \mathbf{z}_{y_n, \varsigma_n})$  for a sequence  $(y_n, \varsigma_n)$  in the canonical model of a neighborhood of  $(\Sigma_0, \mathbf{z}_0)$ . Each  $(\Sigma_{y_n, \varsigma_n}, \mathbf{z}_{y_n, \varsigma_n})$  comes with a Kähler metric, and we can alter it in a small neighborhood of the marked points so that it becomes a Kähler metric of the corresponding orbicurve defined by  $\xi_n$ . Using this Kähler metric and the Hermitian metric  $h_\omega$  on  $(X, J)$ , we can talk about the norm of the gradient of  $\tilde{f}_n$ , denoted by  $|d\tilde{f}_n|$ .

In [FO], the bubbling phenomena is naturally interpreted as a result of degeneration of the domain of the map, which can be easily generalized to the present case, and is achieved in the following lemma in which  $\mu > 0$  is any fixed sufficiently small number.

**Lemma 2.3.12:** *(cf. Proposition 11.3 in [FO]) By increasing the number of marked points and by taking a subsequence if necessary, we may assume that*

$$\sup_{\Sigma_{y_n, \varsigma_n} \setminus W_n(\mu)} |d\tilde{f}_n| < C$$

where  $C$  is independent of  $n$ .



**Lemma 2.3.13:** *Let  $(\Sigma, \mathbf{z}, \mathbf{m})$  be a compact complex orbicurve (with or without boundary) with a compact family of complex structures  $j_n$  and Kähler metrics  $\mu_n$ . Let  $\tilde{f}_n$  be a sequence of  $C^\infty$  maps from  $(\Sigma, \mathbf{z}, \mathbf{m})$  to  $(X, J, \omega)$  which induces a pseudo-holomorphic map  $f_n$  with respect to  $j_n$ , and measured with  $\mu_n$ , the norm of the gradient of  $\tilde{f}_n$ ,  $|d\tilde{f}_n|$ , is uniformly bounded by a constant  $C$ . Then there is a subsequence of  $\tilde{f}_n$ , denoted by  $\tilde{f}_{n_i}$ , such that the corresponding complex structure  $j_{n_i}$  converges to a complex structure  $j_0$ , and  $\tilde{f}_{n_i}$  converges in  $C^\infty$  topology to a  $C^\infty$  map  $\tilde{f}_0$  which induces a pseudo-holomorphic map  $f_0$  with respect to the complex structure  $j_0$ .*

**Proof:** For any  $z \in \Sigma$ , suppose a subsequence  $f_{n_i}(z)$  converges to a point  $p \in X$ . Let  $(V_p, G_p, \pi_p)$  be a local chart at  $p$  with  $U_p = \pi_p(V_p)$ . Then since  $|d\tilde{f}_n|$  is bounded by  $C$ , there exists a disc neighborhood  $D$  such that for large  $n_i$ ,  $\tilde{f}_{n_i}(D)$  lies in  $U_p$ . The issue here is to construct local liftings of  $f_{n_i}$  from the uniformizing system of a fixed neighborhood of  $z$  into  $V_p$ . We may assume that  $D \setminus \{z\}$  contains no orbifold points. For each  $f_{n_i}$ , there is a disc neighborhood  $D_i$  of  $z$ , a branched covering map  $br$  (may be trivial) and a local lifting  $\tilde{f}_i$  into  $V_p$  such that  $\pi_p \circ \tilde{f}_i = f_{n_i} \circ br$ , and for any other point in  $D$ ,  $f_{n_i}$  is lifted to a map into  $V_p$ . It is easily seen that these local liftings can be patched together to define a lifting  $\tilde{f}_{n_i, z}$  on a branched cover of  $D$  into  $V_p$ . The lemma is then reduced to the classical case.  $\square$

**Lemma 2.3.14:** *Let  $(\Sigma, \mathbf{z}, \mathbf{m})$  be a compact complex orbicurve (with or without boundary) with a family of complex structures  $j_n$ . Let  $(\tilde{f}_n, \xi_n)$  be a sequence of good  $C^\infty$  maps from  $(\Sigma, \mathbf{z}, \mathbf{m})$  to  $(X, J, \omega)$ , with isomorphism class of compatible systems  $\xi_n$ , which induces a pseudo-holomorphic map  $f_n$  with respect to  $j_n$ . Assume  $j_n$  converges to a complex structure  $j_0$ , and  $\tilde{f}_n$  converges in the  $C^\infty$  topology to a  $C^\infty$  map  $\tilde{f}_0$  which induces a pseudo-holomorphic map  $f_0$  with respect to  $j_0$ . Then  $\tilde{f}_0$  is good and there is an isomorphism class of compatible systems  $\xi_0$  of  $\tilde{f}_0$  such that a subsequence of  $(\tilde{f}_n, \xi_n)$  converges in the  $C^\infty$  topology to  $(\tilde{f}_0, \xi_0)$ . Moreover, for sufficiently large  $n$ , there exists a  $C^\infty$  section  $\tilde{s}_n$  of the pull-back orbifold bundle  $(TX)_{\xi_0}^*$  over  $(\Sigma, \mathbf{z}, \mathbf{m})$  determined by  $\xi_0$ , such that  $\tilde{f}_n$  equals  $(\tilde{f}_0)_{\xi_0, s_n}$  and  $\xi_n$  equals the canonically determined isomorphism class of compatible systems of  $(\tilde{f}_0)_{\xi_0, s_n}$ . As a consequence,  $(\tilde{f}_0, \xi_0)$  is the unique limit of the said subsequence of  $(\tilde{f}_n, \xi_n)$ .*

**Proof:** First of all, as in the construction of induced geodesic compatible systems, we can construct a compatible cover  $\mathcal{U}$  of  $(\Sigma, \mathbf{z}, \mathbf{m})$  consisting of geodesic discs, a set of geodesic neighborhoods  $\mathcal{U}'_0$  in  $X$ , and a collection of local liftings  $\tilde{f}_{0, U'U'_0}$  of  $f_0$  such that the elements of  $\mathcal{U}$  are in 1:1 correspondence with those of  $\mathcal{U}'_0$ , and if  $U_2 \subset U_1$ , there is an injection between the uniformizing systems of  $U'_{2,0}$  and  $U'_{1,0}$ . Likewise, there is a set of uniformized open subsets  $\mathcal{U}'_n$  for each  $n$ , and a local lifting  $\tilde{f}_{n, UU'_n}$  of  $f_n$  such that for each  $U \in \mathcal{U}$ , there is  $n(U) > 0$ , when  $n > n(U)$ , there is an injection  $\delta_{U'_n}$  from the uniformizing systems of  $U'_n$  into that of  $U'_0$ . There is a 1:1 correspondence between elements of  $\mathcal{U}$  and  $\mathcal{U}'_n$  such that if  $U_2 \subset U_1$ , there is an injection between the uniformizing systems of  $U'_{2,n}$  and  $U'_{1,n}$ . We want to point out that there is a delicate point here. For any collection of finitely many  $C^\infty$  maps, we can construct the sets  $\mathcal{U}$  and  $\mathcal{U}'_n$  with  $\mathcal{U}$  common and  $\mathcal{U}'_n$  consisting of geodesically convex and star-shaped open subsets without any problem, but for infinitely many  $C^\infty$  maps we can not ensure that  $\mathcal{U}'_n$  consists of geodesically convex and star-shaped open subsets while  $\mathcal{U}$  is common to all the maps. This is because there is no lower bound of the injectivity radius of points in  $X$ , although  $X$  is compact. In the present case, we have to exploit the fact that  $\tilde{f}_n$  converges to  $\tilde{f}_0$  and  $f_n$  is pseudo-holomorphic. For each point  $z$  in  $\Sigma$ , there are two possibilities: the first one is that there is a disc neighborhood  $D$  of  $z$  such that, for large  $n$   $f(z)$  has the largest isotropy subgroup amongst the points in  $f_n(D)$ ; the second one is that there is a disc neighborhood  $D$  of  $z$  such that if let  $z_n \in D$  be a point such that  $f_n(z_n)$  has the largest isotropy subgroup amongst the points in  $f_n(D)$ , then  $z_n \neq z$  and  $z_n \rightarrow z$  as  $n \rightarrow \infty$ . In the first case, we can pick a geodesically convex and star-shaped open subset centered at  $f_n(z)$  (not necessarily a ball) for the preliminary

candidate of  $U'_n$ , while in the second case, we pick a geodesically convex and star-shaped open subset centered at  $f_n(z_k)$ . One can verify that this will work out.

Now we use the given isomorphism class of compatible systems  $\xi_n$  to define, for each injection  $i$  from the uniformizing system of  $U_2$  into that of  $U_1$ , an injection  $\lambda_n(i)$  from the uniformizing system of  $U'_{2,n}$  to that of  $U'_{1,n}$  such that  $\{\tilde{f}_n, \lambda_n(i)\}$  becomes a compatible system of  $\tilde{f}_n$  within the isomorphism class of  $\xi_n$ . We first pick a compatible system  $\{\tilde{f}_n, \lambda_n(i)\}$  in  $\xi_n$ . For each  $U \in \mathcal{U}$  such that  $U \subset \Sigma \setminus \mathbf{z}$ , we pick a  $W_n \in \mathcal{W}_n$  such that  $W_n \subset U$  and there is an injection  $\delta_{W_n U}$  from the uniformizing system of  $W'_n$  into that of  $U'_n$  satisfying  $\delta_{W_n U} \circ \tilde{f}_n|_{W'_n} = \tilde{f}_n|_{U'_n} \circ \delta_{W_n U}$ . We will fix  $\delta_{W_n U}$ . Then this can be extended out to determine an injection  $\delta_{W_{1,n} U}$  for any other element  $W_{1,n} \in \mathcal{W}_n$  such that  $W_{1,n} \subset U$  whenever it is possible. Now for any  $U_1, U_2 \in \mathcal{U}$  with an inclusion  $i : U_2 \subset U_1$ , we take a  $W_n \in \mathcal{W}_n$ ,  $W_n \subset U_2 \subset U_1$ , and we define  $\lambda_n(i)$  to be the unique injection such that  $\delta_{W_n U_2} = \lambda_n(i) \circ \delta_{W_n U_1}$ . One can check that  $\{\tilde{f}_n, \lambda_n(i)\}$  is a compatible system of  $\tilde{f}_n$  on  $\Sigma \setminus \mathbf{z}$ , which extends over  $\Sigma$  uniquely, and is in the isomorphism class of  $\xi_n$ .

Now we take a sequence of subsets  $\mathcal{U}_k$  of  $\mathcal{U}$  such that  $\mathcal{U}_k \subset \mathcal{U}_{k+1}$ ,  $\mathcal{U} = \cup_k \mathcal{U}_k$ , and each  $\mathcal{U}_k$  is a finite cover of  $\Sigma$ . By the finiteness of each  $\mathcal{U}_k$ , there is a subsequence of  $\{\tilde{f}_n, \lambda_n(i)\}$  such that for any injection  $i$  associated to an inclusion  $U_2 \subset U_1$ ,  $U_1, U_2 \in \mathcal{U}_k$ , we have an injection  $\lambda_0(i)$  such that  $\delta_{U'_{1,n}} \circ \lambda_n(i) = \lambda_0(i) \circ \delta_{U'_{2,n}}$  and  $\delta_{U'_n} \circ \tilde{f}_n|_{U'_n}$  converges to  $\tilde{f}_0|_{U'_0}$ . Then it follows that the diagonal subsequence converges to  $\{\tilde{f}_0, \lambda_0(i)\}$ .

Finally, we take a finite subset  $\mathcal{U}_0$  of  $\mathcal{U}$  such that  $\mathcal{U}_0$  covers  $\Sigma$  and for any  $U_1, U_2 \in \mathcal{U}_0$ , there is a  $U_3 \in \mathcal{U}_0$  such that  $U_3 \subset U_1 \cap U_2$ . Then for each  $n > N = \max_{U \in \mathcal{U}_0} n(U)$ , the image of  $\delta_{U'_n} \circ \tilde{f}_n|_{U'_n}$  lies in the uniformizing system of  $U'_0$  for any  $U \in \mathcal{U}_0$ . This gives rise to a collection of local  $C^\infty$  sections  $\tilde{s}_U$  over  $U$  of the pull-back orbifold bundle  $(TX)_{\xi_0}^*$  defined by  $\{\tilde{f}_0, \lambda_0(i)\}$ . On the intersection  $U_1 \cap U_2$ ,  $\tilde{s}_{U_1} = \tilde{s}_{U_2}$  restricted to  $U_3$ . By the unique continuity property of pseudo-holomorphic maps,  $\tilde{s}_{U_1} = \tilde{s}_{U_2}$  on  $U_1 \cap U_2$ , so that they patch together to a global section  $\tilde{s}_n$ , and  $\tilde{f}_n$  equals  $(\tilde{f}_0)_{\xi_0, s_n}$ . The fact that  $\xi_n$  equals the canonically determined isomorphism class of compatible systems of  $(\tilde{f}_0)_{\xi_0, s_n}$  follows from the description of the difference of any two isomorphism classes of compatible systems of  $\tilde{f}_n$  in terms of a homomorphism from  $\pi_1(\Sigma \setminus \mathbf{z}, z_0)$  to  $G_{f_n(z_0)}$ .  $\square$ .

Now for a closed symplectic orbifold  $(X, \omega)$  with a  $\omega$ -compatible almost complex structure  $J$ , the argument in [FO] can be taken word by word to show that the moduli space  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  is compact, except that we have to make sure that a limiting compatible system is determined at each nodal point of the limiting curve and the convergence is as specified in Definition 2.3.7. But this follows from Lemma 2.2.4, Lemma 2.2.6 (see the remark after the proof of Lemma 2.2.6) easily.

Now let's consider the case when  $(X, J)$  is a projective orbifold with a Hermitian metric  $h$  on it. By assumption,  $X \subset \mathbf{P}^N$  is a subvariety of a projective space  $\mathbf{P}^N$ . A continuous map  $f : \Sigma \rightarrow X$  is analytic if and only if  $f : \Sigma \rightarrow \mathbf{P}^N$  is holomorphic. Now suppose we have a sequence of stable maps  $(f_n, (\Sigma_n, \mathbf{z}_n), \xi_n)$  in  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ ; forgetting the twisted boundary conditions  $\xi_n$ , we have a sequence of stable maps  $(f_n, (\Sigma_n, \mathbf{z}_n))$  in  $\overline{\mathcal{M}}_{g,k}(\mathbf{P}^N, J_0, A)$ , and by the classical Gromov compactness theorem, there is a subsequence converging to a stable map  $(f_0, (\Sigma_0, \mathbf{z}_0))$  in  $\overline{\mathcal{M}}_{g,k}(\mathbf{P}^N, J_0, A)$ , which is measured by the Fubini-Study metric on  $\mathbf{P}^N$ . Let's still use  $(f_n, (\Sigma_n, \mathbf{z}_n))$  to denote the subsequence, and assume that additional marked points have been added. Then for sufficiently large  $n$ , there are points  $(y_n, \varsigma_n)$  in the canonical model of a neighborhood of  $(\Sigma_0, \mathbf{z}_0)$  in the Deligne-Mumford compactification such that  $(\Sigma_n, \mathbf{z}_n)$  are identified with  $(\Sigma_{y_n, \varsigma_n}, \mathbf{z}_{y_n, \varsigma_n})$ , and for any small  $\mu > 0$ , the restriction of  $f_n$  to  $\Sigma_{y_n, \varsigma_n} \setminus W_n(\mu)$  converges to  $f_0$  in  $C^\infty$ , and  $\lim_{\mu \rightarrow 0} \sup \lim_{n \rightarrow \infty} \text{Diam}(W_n(\mu)) = 0$ . Here the  $C^\infty$  norm and the distance function are defined from the Fubini-Study metric. We observe:

- Lemma 2.3.11 (cf. Lemma 11.2 in [FO]) holds for general Hermitian orbifolds.

- The convergence of  $f_n$  to  $f_0$  on  $\Sigma_{y_n, \varsigma_n} \setminus W_n(\mu)$  (although measured by the Fubini-Study metric) implies that

$$\sup_{\Sigma_{y_n, \varsigma_n} \setminus W_n(\mu)} |d\tilde{f}_n|_h < C$$

where  $|\cdot|_h$  is the norm defined by the Hermitian metric  $h$  on  $X$ , and  $C$  is independent of  $n$ , for otherwise, a bubble would have developed, which contradicts the convergence of  $f_n$  to  $f_0$  in  $\mathbf{P}^N$ .

- $\lim_{\mu \rightarrow 0} \sup \lim_{n \rightarrow \infty} \text{Diam}_h(W_n(\mu)) = 0$  holds where  $\text{Diam}_h$  is the diameter defined by  $h$ , since otherwise a bubble would have developed.

With these understood, a similar argument shows that the moduli space  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  is compact for a projective orbifold  $(X, J)$ , measured by any fixed Hermitian metric  $h$ .

Finally, we will show that the moduli space  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  is metrizable. We recall the Metrization Theorem of Urysohn (see [Ke]):

*A topological space  $X$  is metrizable if the following conditions hold:*

- $\{x\}$  is closed for each point  $x \in X$ , i.e.,  $X$  is a  $T_1$ -space.
- For each  $x \in X$  and each neighborhood  $U$  of  $x$ , there is a closed neighborhood  $V$  of  $x$  such that  $V \subset U$  (i.e.  $X$  is regular).
- $X$  has a countable base.

First, the moduli space  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  is Hausdorff, as shown in [FO], so it is a  $T_1$ -space.

Secondly, for each point  $x_0$  in  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ , represented by a stable map  $(f_0, (\Sigma_0, \mathbf{z}_0), \xi_0)$ , we will define a pseudo-distance function  $d$  as follows. For any point  $x = [(f, (\Sigma, \mathbf{z}), \xi)]$  in  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ , let  $d_1(x_0, x) = \min(\text{dist}((\Sigma_0, \mathbf{z}_0)^+, (\Sigma, \mathbf{z})^+))$  where  $(\Sigma, \mathbf{z})^+$  denotes any stabilization of  $(\Sigma, \mathbf{z})$ , and  $\text{dist}$  is the distance function on the corresponding Deligne-Mumford compactification; let  $d_2(x_0, x) = \min_{f: [f] \in x} \max_{f^+} \max_{(z, z_0) \in (\mathbf{z}^+, \mathbf{z}_0^+)} (\text{dist}_h(f^+(z), f_0^+(z_0)))$ , and let  $d = d_1 + d_2$ . Then fixing a small  $\mu > 0$ , there is an  $\epsilon(x_0) > 0$  such that if  $d(x_0, x) \leq \epsilon(x_0)$ , there is a representative  $(f, (\Sigma, \mathbf{z}), \xi)$  of  $x$  such that the restriction of  $\tilde{f}$  to  $\Sigma \setminus W(\mu)$  equals  $(\tilde{f}_0)_{\xi'_0, s}$  for some  $\xi'_0$  of  $f_0$  and a  $C^\infty$  section  $\tilde{s}$ . Now for any  $0 < \epsilon \leq \epsilon(x_0)$ , we define the set  $U(x_0, \epsilon)$  to be the set of all  $x \in \overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  such that  $d(x_0, x) < \epsilon$  and if a representative  $(f, (\Sigma, \mathbf{z}), \xi)$  of  $x$  equals  $(\tilde{f}_0)_{\xi'_0, s}$  on  $\Sigma \setminus W(\mu)$ , then  $\xi'_0 = \xi_0$ . Then  $U(x_0, \epsilon)$  defines a family of open neighborhoods of  $x_0$  and  $\cap_\epsilon U(x_0, \epsilon) = \{x_0\}$ . Observe that for any  $\epsilon_1 < \epsilon$ ,  $\overline{U(x_0, \epsilon_1)} \subset U(x_0, \epsilon)$ , hence  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  is regular.

Finally,  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  has a countable base, since it is compact, and at each point  $x_0$ , there is a family of open neighborhoods  $U(x_0, \epsilon)$  of  $x_0$  where  $0 < \epsilon \leq \epsilon(x_0)$ . Hence the moduli space  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  is metrizable by Urysohn's theorem.

### 3 Virtual Cycle and Orbifold Gromov-Witten Invariants

The orbifold Gromov-Witten invariants are defined by constructing the virtual fundamental cycle on the moduli spaces of orbifold stable maps using the technique of Li-Tian [LT] and Fukaya-Ono [FO]. For this purpose, we first construct a Kuranishi structure on the moduli space along the lines of [FO], although our treatment of the analysis is different. By a general procedure in [FO], the Kuranishi structure already gives rise to a virtual fundamental class, which suffices to define the GW invariants and prove its usual axioms.

### 3.1 Review of Kuranishi structure

In this subsection, we give a brief review of the notion of Kuranishi structure; see [FO] for more details.

Let  $X$  be a compact, metrizable topological space.

**Definition 3.1.1:** A Kuranishi neighborhood of  $p \in X$  is a system  $(U_p, E_p, s_p, \psi_p)$  where

- $U_p = V_p/G_p$  is an orbifold and  $E_p$  is an orbifold bundle on it.
- $s_p$  is a continuous section of  $E_p$ .
- $\psi_p$  is a homeomorphism from  $s_p^{-1}(0)$  to a neighborhood of  $p$  in  $X$ .

We call  $E_p$  the obstruction bundle and  $s_p$  the Kuranishi map.

For any  $q \in \text{Im}\psi_p$ , the Kuranishi neighborhood  $(U_p, E_p, s_p, \psi_p)$  induces a Kuranishi neighborhood of  $q$  by restricting to a neighborhood of  $\psi_p^{-1}(q)$  in  $U_p$ . Two Kuranishi neighborhoods of  $p$ ,  $(U_p, E_p, s_p, \psi_p)$  and  $(U'_p, E'_p, s'_p, \psi'_p)$ , are *isomorphic* if there exists an isomorphism  $(I, J) : (U_p, E_p) \rightarrow (U'_p, E'_p)$  such that  $J \circ s_p = s'_p \circ I$  and  $\psi'_p \circ I = \psi_p$ . A *germ* of Kuranishi neighborhood of  $p$  is defined in the following sense:  $(U_p, E_p, s_p, \psi_p)$  and  $(U'_p, E'_p, s'_p, \psi'_p)$  are *equivalent* if they induce isomorphic Kuranishi neighborhoods of  $p$  on a smaller neighborhood of  $p$ . A germ of Kuranishi neighborhood is said to be a *stabilization* of another if there are representatives  $(U'_p, E'_p, s'_p, \psi'_p)$  and  $(U_p, E_p, s_p, \psi_p)$  respectively and an embedding of orbifolds  $(\phi, \hat{\phi}) : (U_p, E_p) \rightarrow (U'_p, E'_p)$  such that  $s'_p \circ \phi = \hat{\phi} \circ s_p$  and  $\psi'_p \circ \phi = \psi_p$ , together with an isomorphism  $\Phi : (TU'_p)^*/TU_p \rightarrow E'_p/E_p$  of orbibundles on  $U_p$ , where  $(TU'_p)^*/TU_p$  is the normal bundle of  $\phi(U_p)$  in  $U'_p$ . One can similarly define the notion of *germ* of  $(\phi, \hat{\phi}, \Phi)$ .

**Definition 3.1.2:** A Kuranishi structure of dimension  $n$  on  $X$  assigns a germ of Kuranishi neighborhood to each point  $p \in X$ , such that for each representative  $(U_p, E_p, s_p, \psi_p)$  of it,  $\dim U_p - \text{rank } E_p = n$ , and the induced germ of Kuranishi neighborhood at any  $q \in \text{Im}\psi_p$  is a stabilization of the germ of Kuranishi neighborhood of  $q$ . Moreover, for each sufficiently small representative  $(U_q, E_q, s_q, \psi_q)$ , there is a  $(\phi_{pq}, \hat{\phi}_{pq}, \Phi_{pq}) : (U_q, E_q, (TU_q)^*/TU_q) \rightarrow (U_p, E_p, E_p/E_q)$  satisfying the following compatibility condition: for each  $r \in \text{Im}\psi_q \cap \text{Im}\psi_p$  and any sufficiently small representative  $(U_r, E_r, s_r, \psi_r)$ , the equation

$$(\phi_{pq}, \hat{\phi}_{pq}) \circ (\phi_{qr}, \hat{\phi}_{qr}) = (\phi_{pr}, \hat{\phi}_{pr})$$

holds and the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & (TU_q)^*/TU_r & \rightarrow & (TU_p)^*/TU_r & \rightarrow & (TU_p)^*/TU_q|_{U_r} \rightarrow 0 \\ & & \downarrow \Phi_{qr} & & \downarrow \Phi_{pr} & & \downarrow \Phi_{pq} \\ 0 & \rightarrow & E_q/E_r & \rightarrow & E_p/E_r & \rightarrow & E_p/E_q|_{U_r} \rightarrow 0 \end{array}$$

commutes.

**Remark 3.1.3:** A germ of triple  $(\phi_{pq}, \hat{\phi}_{pq}, \Phi_{pq})$  is well determined in the above definition for any sufficiently two close points  $p$  and  $q$  in  $X$ , with a suitable compatibility condition satisfied. The point of imposing the compatibility condition here is that although a Kuranishi structure is not a system of trivializations of an orbifold bundle, but defines a certain abstract object, a sort of quasi-orbifold bundle such that  $X$  is the zero set of a continuous section of it.

A  $K$ -theory can be defined over Kuranishi structures. We first give a definition of an analogue of vector bundles over a space with a Kuranishi structure.

**Definition 3.1.4:** Let  $X$  be a space with a Kuranishi structure. A bundle system on  $X$  consists of the following data:

- For each point  $p \in X$  there exists a germ of a pair of orbifold bundles  $(F_{1,p}, F_{2,p})$  on its Kuranishi neighborhood.
- For two sufficiently close points  $p$  and  $q$  in  $X$ , for which the germ of  $(\phi_{pq}, \hat{\phi}_{pq}, \Phi_{pq})$  is defined, there exist a germ of embeddings  $(\Psi_{1,pq}, \Psi_{2,pq}) : (F_{1,q}, F_{2,q}) \rightarrow (F_{1,p}, F_{2,p})$  and a germ of isomorphisms of orbibundles  $\Psi_{pq} : F_{1,p}|_{U_q}/F_{1,q} \rightarrow F_{2,p}|_{U_q}/F_{2,q}$ .
- The following compatibility condition holds: for any triple  $(r, q, p)$  such that the germs of  $(\phi_{qr}, \hat{\phi}_{qr}, \Phi_{qr})$ ,  $(\phi_{pr}, \hat{\phi}_{pr}, \Phi_{pr})$  and  $(\phi_{pq}, \hat{\phi}_{pq}, \Phi_{pq})$  are defined, the equation

$$(\Psi_{1,pq}, \Psi_{2,pq}) \circ (\Psi_{1,qr}, \Psi_{2,qr}) = (\Psi_{1,pr}, \Psi_{2,pr})$$

holds and the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & F_{1,q}|_{U_r}/F_{1,r} & \rightarrow & F_{1,p}|_{U_r}/F_{1,r} & \rightarrow & F_{1,p}|_{U_r}/F_{1,q}|_{U_r} \rightarrow 0 \\ & & \downarrow \Psi_{qr} & & \downarrow \Psi_{pr} & & \downarrow \Psi_{pq} \\ 0 & \rightarrow & F_{2,q}|_{U_r}/F_{2,r} & \rightarrow & F_{2,p}|_{U_r}/F_{2,r} & \rightarrow & F_{2,p}|_{U_r}/F_{2,q}|_{U_r} \rightarrow 0 \end{array}$$

commutes.

One can define Whitney sum, tensor product, etc. of bundle systems in an obvious way, as well as an isomorphism relation between bundle systems.

**Example 3.1.5:** It is easily seen that the germ of pairs  $(TU_p, E_p)$  defines a bundle system; we call it the tangent bundle of  $X$ , and denote it by  $TX$ .

A bundle system is said to be *oriented* if  $F_{1,p}, F_{2,p}$  are oriented for each  $p \in X$  and  $\Psi_{pq} : F_{1,p}|_{U_q}/F_{1,q} \rightarrow F_{2,p}|_{U_q}/F_{2,q}$  is orientation preserving. It is said to be *complex* if  $F_{1,p}, F_{2,p}$  are complex and  $\Psi_{1,pq}, \Psi_{2,pq}$  and  $\Psi_{pq}$  are complex linear. A bundle system  $((F_{1,p}, F_{2,p}), (\Psi_{1,pq}, \Psi_{2,pq}, \Psi_{pq}))$  is said to be *trivial* if there exist germs of isomorphisms  $F_{1,p} \cong F_{2,p}$  which are compatible with  $(\Psi_{1,pq}, \Psi_{2,pq}, \Psi_{pq})$ .

Consider the free abelian group generated by the set of all isomorphism classes of bundle systems and divide it by the relations

$$\begin{aligned} & [((F_{1,p}, F_{2,p}), (\Psi_{1,pq}, \Psi_{2,pq}, \Psi_{pq})) \oplus ((F'_{1,p}, F'_{2,p}), (\Psi'_{1,pq}, \Psi'_{2,pq}, \Psi'_{pq}))] \\ &= [((F_{1,p}, F_{2,p}), (\Psi_{1,pq}, \Psi_{2,pq}, \Psi_{pq}))] + [((F'_{1,p}, F'_{2,p}), (\Psi'_{1,pq}, \Psi'_{2,pq}, \Psi'_{pq}))] \\ & [((F_{1,p}, F_{2,p}), (\Psi_{1,pq}, \Psi_{2,pq}, \Psi_{pq}))] = 0 \text{ if } ((F_{1,p}, F_{2,p}), (\Psi_{1,pq}, \Psi_{2,pq}, \Psi_{pq})) \text{ is trivial.} \end{aligned}$$

This group is called the real  $K$ -group of  $X$  with the Kuranishi structure, and is denoted by  $KO(X)$ . By using oriented bundle systems and complex bundle systems, the corresponding  $K$ -groups  $KSO(X)$  and  $K(X)$  are defined, and there is an obvious map

$$K(X) \rightarrow KSO(X) \rightarrow KO(X).$$

A Kuranishi structure  $X$  is said to be *stably orientable* if  $[TX]$  is in the image of  $KSO(X)$ . It is said to be *stably complex* if  $[TX]$  is in the image of  $K(X)$ . It is proved in [FO] that being stably orientable is equivalent to being orientable. It is obvious that if  $X$  is stably complex, then  $X$  is orientable with a canonical orientation.

### 3.2 Construction of Kuranishi neighborhood

This subsection is devoted to a local construction of Kuranishi neighborhood for a point in the moduli space of stable maps  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ . First we review briefly the basic idea of the construction so that for those readers who are unfamiliar with this type of argument, the big picture will not be buried in a sea of technical details.

The construction of Kuranishi model for a nonlinear Fredholm map is based on the following simple lemma of Banach. Let  $B : U \rightarrow U$  be a map from a ball of radius  $r$  in a Banach space to itself such that

$$\|B(x) - B(y)\| \leq \epsilon \|x - y\|$$

holds for any  $x, y \in U$  for an  $\epsilon < 1$ , then the map  $B$  has a unique fixed point  $x_0$  in  $U$ , i.e.,  $x_0 = B(x_0)$ . Moreover, if  $B$  depends on a parameter, say  $u$ , and

$$\|B_u(x) - B_u(y)\| \leq \epsilon \|x - y\|$$

holds for all  $u$  for an  $\epsilon < 1$ , then letting  $x_u$  be the fixed point of  $B_u$ , we have

$$\|x_{u_1} - x_{u_2}\| \leq \frac{1}{1 - \epsilon} \|B_{u_1}(x_{u_2}) - B_{u_2}(x_{u_2})\|,$$

and when  $\|B_{u_1}(x) - B_{u_2}(x)\| \leq C \|u_1 - u_2\| \|x\|$  for any  $x \in U$  for some constant  $C > 0$ , we have

$$\|x_{u_1} - x_{u_2}\| < \|u_1 - u_2\|$$

when we restrict the maps  $B_u$  to a smaller ball of radius less than  $\frac{1-\epsilon}{C}$ .

Now suppose  $F : X \rightarrow Y$  is a  $C^2$  nonlinear Fredholm map from a neighborhood of 0 in a Banach space  $X$  into a neighborhood of 0 in a Banach space  $Y$  such that  $F(0) = 0$ . Let  $L$  be the differential of  $F$  at 0, which is a linear Fredholm map from  $X$  to  $Y$ . Let  $E$  be a finite dimensional subspace of  $Y$  such that  $Y$  is spanned by  $E$  and the image of  $L$ ,  $ImL$ . Let  $V = \{x \in X | Lx \in E\}$ ; then  $V$  is a finite dimensional subspace in  $X$ ,  $\dim V - \dim E = index L$ , and there exist subspaces  $V'$  and  $E'$  of  $X$  and  $Y$  such that  $X = V \oplus V'$ ,  $Y = E' \oplus E$ ,  $L : V' \rightarrow E'$  is invertible, and if  $x = v + v'$  for  $v \in V$ ,  $v' \in V'$ ,  $y = e' + e$  for  $e' \in E'$ ,  $e \in E$ , then  $\|v\| \leq c\|x\|$ ,  $\|v'\| \leq c\|x\|$  and  $\|e'\| \leq c\|y\|$ ,  $\|e\| \leq c\|y\|$  hold for some constant  $c > 0$ . Write  $F(x) = Lx + G(x)$ ; then  $G(0) = 0$  and  $DG(0) = 0$ . (Here  $DG$  stands for the differential of  $G$ .) Let's consider solving the equation  $F(x) \in E$ . If we write  $x = v + v'$ , and  $PG(x)$  for the component of  $G(x)$  in  $E'$ , the equation  $F(x) \in E$  becomes  $0 = Lv' + PG(v + v')$ . If we let  $B_v(x) = -L^{-1}(PG(v + x))$ , then the above equation becomes a fixed point equation for  $B_v$ , i.e.,  $v' = B_v(v')$ . Let's examine the norms  $\|B_v(x_1) - B_v(x_2)\|$  and  $\|B_{v_1}(x) - B_{v_2}(x)\|$ .

$$\begin{aligned} \|B_v(x_1) - B_v(x_2)\| &\leq \|L^{-1}\| \|PG(v + x_1) - PG(v + x_2)\| \\ &\leq c \|L^{-1}\| \|G(v + x_1) - G(v + x_2)\| \\ &\leq c \|L^{-1}\| \|DG(v + t_0x_1 + (1 - t_0)x_2)\| \|x_1 - x_2\|, \end{aligned}$$

and similarly,

$$\|B_{v_1}(x) - B_{v_2}(x)\| \leq c \|L^{-1}\| \|DG(x + t_0v_1 + (1 - t_0)v_2)\| \|v_1 - v_2\|.$$

**Lemma 3.2.1:** *Consider the ball  $U_{2r}$  in  $X$  of radius  $2r$  such that  $x \in U_{2r}$  satisfies the condition  $c\|L^{-1}\| \|DG(x)\| \leq \frac{1}{3}$ . Then for any  $v \in V$  such that  $v \in U_r$ , there is a unique  $v'(v) \in V' \cap U_r$*

such that  $F(v + v'(v)) \in E$ , and  $\psi : v \rightarrow v + v'(v)$  is 1 : 1. On the other hand, for any  $x \in U_{r/c}$  such that  $F(x) \in E$ , there is a unique  $v \in V \cap U_r$  such that  $x = v + v'(v)$ . In particular, let  $f : V \cap \psi^{-1}(U_{r/c}) \rightarrow E$  be defined by  $f(v) = F(v + v'(v))$ , then  $f$  is continuous and the zero set  $f^{-1}(0)$  is homeomorphic to the zero set  $F^{-1}(0) \cap U_{r/c}$  by  $v \rightarrow v + v'(v)$ .

**Proof:** The condition  $c\|L^{-1}\| \|DG(x)\| \leq \frac{1}{3}$  for all  $x \in U_{2r}$  implies that if  $v \in V \cap U_r$ ,  $x \in V' \cap U_r$ , then  $B_v(x) \in V' \cap U_r$ , and the inequality

$$\|B_v(x_1) - B_v(x_2)\| \leq \frac{1}{3}\|x_1 - x_2\|$$

holds. So for any  $v \in V \cap U_r$ , there is a unique fixed point  $v'(v) \in V' \cap U_r$  of  $B_v$ , so that  $F(v + v'(v)) \in E$ . That the map  $v \rightarrow v + v'(v)$  is 1 : 1 follows from the inequality

$$\|B_{v_1}(x) - B_{v_2}(x)\| \leq \frac{1}{3}\|v_1 - v_2\|$$

which implies that  $\|v'(v_1) - v'(v_2)\| \leq \frac{2}{3}\|v_1 - v_2\|$ .

On the other hand, if  $x \in U_{r/c}$  and  $F(x) \in E$ , and if we write  $x = v + v'$ , we have  $v, v' \in U_r$ , and  $v' = B_v(v')$ . Hence  $x$  lies in the image of  $\psi : v \rightarrow v + v'(v)$ .  $\square$

**Remark 3.2.2:** In this construction, the solution set of  $F(x) \in E$  in the ball  $U_{r/c}$  of radius  $r/c$  is identified with an open set  $\psi^{-1}(U_{r/c}) \cap V \cap U_r$  in a finite dimensional ball of radius  $r$ , and the zero set  $F^{-1}(0) \cap U_{r/c}$  is identified with the zero set of a continuous function  $f : \psi^{-1}(U_{r/c}) \cap V \cap U_r \rightarrow E$  between finite dimensional spaces. This is generally referred to as the Kuranishi model of the map  $F$  at 0. The size of the Kuranishi model, which is determined by the number  $r$ , can be explicitly measured by the constant  $c$ , the norm  $\|L^{-1}\|$  and the second derivative  $D^2F(0)$  of  $F$  at 0, while the constant  $c$  depends solely on the relative positions of the subspace  $E$  and  $\text{Im } L$  in  $Y$ . (We will call the constant  $c$  the ratio of the corresponding decompositions of Banach spaces  $X$  and  $Y$ .) This knowledge is important in applications because very often we need to carry out a parametrized version of this construction, and we need to know how the size of the Kuranishi model is changing with the parameter.

In the construction of Kuranishi neighborhood for a stable map, we encounter a family of nonlinear Fredholm maps  $F_s : X_s \rightarrow Y_s$  parametrized by a finite dimensional ball  $s \in B$ , such that  $F_0(0) = 0$  and  $F_s(0)$  is small. There are no explicit identifications between the  $X_s$ 's and the  $Y_s$ 's. We will find a family of isomorphisms  $\theta_s : E_0 \rightarrow E_s$  and  $\eta_s : V_0 \rightarrow V_s$  such that  $h_s = (\theta_s)^{-1} \circ f_s \circ \eta_s$  is continuous in  $s$  so that we can construct a Kuranishi neighborhood  $f : B \times V_0 \rightarrow E_0$  by  $(s, v) \rightarrow h_s(v)$ . We need to watch out for the dependence of  $c_s$ ,  $\|L_s^{-1}\|$ ,  $\|D^2F_s(0)\|$  and  $\|\theta_s\|$ ,  $\|\eta_s\|$  on the parameter  $s$ . Moreover, there are finite groups  $G$  and  $H$  acting on the parameter space  $B$  and the vector space  $V_0$  respectively.  $\square$

Now we start out with the construction. Consider the biholomorphism  $\mathbf{R} \times S^1 \rightarrow \mathbf{C}^* \setminus \{0\}$  given by  $t + is \rightarrow e^{-(t+is)}$  where  $s \in S^1 = \mathbf{R}/2\pi\mathbf{Z}$ . This identifies the punctured disc of radius  $r$  with the half-cylinder  $[-\ln r, \infty) \times S^1$ . Let  $f : (\Sigma, \mathbf{z}) \rightarrow (X, J, h)$  be a pseudo-holomorphic map with a twisted boundary condition  $\xi$  from a marked Riemann surface into a Hermitian orbifold (cf. Remark 2.2.10). Let  $z_0 \in \mathbf{z}$  be one of the marked points. We intend to think of  $f$  as a map  $f_0$  from the punctured Riemann surface  $(\Sigma, \mathbf{z}) \setminus \{z_0\}$  into  $X$ , where  $(\Sigma, \mathbf{z}) \setminus \{z_0\}$  is understood as a Riemann surface with a cylindrical end via the above biholomorphism  $\mathbf{R} \times S^1 \rightarrow \mathbf{C}^* \setminus \{0\}$ , as justified by Lemma 3.2.3.

Recall that we let  $(TX)_\xi^*$  denote the pull-back orbifold bundle defined by  $(f, \xi)$ . For any  $C^\infty$  section  $\tilde{s}$  of  $(TX)_\xi^*$ , let  $\tilde{f}_{\xi, s} = \text{Exp} \circ \tilde{f}_\xi \circ \tilde{s}$ ,  $(TX)_{\xi, s}^*$  be the pull-back orbifold bundle defined by  $\tilde{f}_{\xi, s}$ ,

and  $Par_t : (TX)_\xi^* \rightarrow (TX)_{\xi,ts}^*$  be the orbifold bundle isomorphism defined by parallel transport along parametrized geodesics  $\tilde{\gamma}_z(t) = Exp \circ \tilde{f}_\xi \circ t\tilde{s}(z)$  (cf. Lemma 4.4.15). We define a map  $F : L_1^p((TX)_\xi^*) \rightarrow L^p((TX)_\xi^* \otimes \Lambda^{0,1}(\Sigma))$  (for  $p > 2$ ) by  $F(\tilde{s}) = ((Par_1)^{-1} \times Id) \circ \bar{\partial}(\tilde{f}_{\xi,s})$ , where

$$\bar{\partial}(\tilde{f}_{\xi,s}) = \frac{1}{2}(d\tilde{f}_{\xi,s} + J \circ d\tilde{f}_{\xi,s} \circ j) \in C^\infty((TX)_{\xi,s}^* \otimes \Lambda^{0,1}(\Sigma)).$$

Then  $F$  is a nonlinear Fredholm map. Its linearization  $DF_{\tilde{s}}$  is a first order elliptic operator (in the orbifold category) whose index is given by the Hirzebruch-Riemann-Roch formula:

$$index(DF_{\tilde{s}}) = 2c_1(|(TX)_\xi^*|)([\Sigma]) + 2n(1 - g_\Sigma).$$

Here  $2n$  is the dimension of  $X$ , and  $|(TX)_\xi^*|$  is the de-singularization of  $(TX)_\xi^*$ . (cf. Proposition 4.1.4 in [CR1].)

For the corresponding pseudo-holomorphic map  $f_0$  from the Riemann surface with a cylindrical end  $(\Sigma, \mathbf{z}) \setminus \{z_0\}$  into  $X$ , let  $\xi_0$  denote the isomorphism class of compatible systems induced by  $\xi$ ; we have the corresponding pull-back orbibundles  $(TX)_{\xi_0}^*$ ,  $(TX)_{\xi_0,s}^*$ , the  $C^\infty$  maps  $(\tilde{f}_0)_{\xi_0,s}$ , and the orbifold bundle isomorphisms  $Par_t$ . Let  $\delta \in (0, \epsilon_0)$  for a sufficiently small  $\epsilon_0$ . We define a map

$$F_0 : L_{1,\delta}^p((TX)_{\xi_0}^*; (TX)_{f(z_0)}^\xi) \rightarrow L_\delta^p((TX)_{\xi_0}^* \otimes \Lambda^{0,1}(\Sigma \setminus \{z_0\})) \quad (\text{for } p > 2)$$

by  $F_0(\tilde{s}) = ((Par_1)^{-1} \times Id) \circ \bar{\partial}((\tilde{f}_0)_{\xi_0,s})$ , where  $L_{1,\delta}^p((TX)_{\xi_0}^*; (TX)_{f(z_0)}^\xi)$  is the space of local  $L_1^p$  sections of  $(TX)_{\xi_0}^*$  which exponentially decay to an element in  $(TX)_{f(z_0)}^\xi$  with a weight  $\delta$ ,  $(TX)_{f(z_0)}^\xi$  is the linear subspace of fixed points of the group homomorphism of  $\xi$  at  $z_0$  in the fiber  $TX_{f(z_0)}$ . For  $\tilde{s} \in L_{1,\delta}^p((TX)_{\xi_0}^*; (TX)_{f(z_0)}^\xi)$ , let  $s_\infty$  be its limiting value at infinity; we define its norm by

$$\|\tilde{s}\| = \|\tilde{s} - s_\infty\|_{L_{1,\delta}^p} + \|s_\infty\|.$$

**Lemma 3.2.3:** *When restricted to a small neighborhood of the zero section in  $L_{1,\delta}^p((TX)_{\xi_0}^*; (TX)_{f(z_0)}^\xi)$ , the map  $F_0$  is a nonlinear Fredholm map with the same index as  $F$ . Moreover, the zero set of  $F_0$  is identical with the zero set of  $F$ , i.e.,  $F_0^{-1}(0) = F^{-1}(0)$ .*

**Proof:** In order to show that  $F_0$  is Fredholm, we need to look at the restriction of its linearization  $(DF_0)_{\tilde{s}}$  to the cylindrical end. Let  $p = f(z_0)$  and  $(V, G, \pi)$  be a geodesic chart of  $X$  at  $p$ , with complex coordinates  $u^\alpha, \alpha = 1, \dots, n$ . Then in  $V$  the  $\bar{\partial}$  operator is given by

$$\bar{\partial}(u^\alpha) = \frac{1}{2}\left(\frac{\partial u^\alpha}{\partial t} - J(u^\alpha)\frac{\partial u^\alpha}{\partial s}\right)dt + \frac{1}{2}\left(\frac{\partial u^\alpha}{\partial s} + J(u^\alpha)\frac{\partial u^\alpha}{\partial t}\right)ds.$$

The linearization  $(D\bar{\partial})_{u^\alpha}$  at  $(u^\alpha)$  is

$$\begin{aligned} (D\bar{\partial})_{u^\alpha}(v^\alpha) &= \frac{1}{2}\left(\frac{\partial v^\alpha}{\partial t} - J(u^\alpha)\frac{\partial v^\alpha}{\partial s} - (\partial_\beta J)(u^\alpha)\frac{\partial u^\alpha}{\partial s}v^\beta\right)dt \\ &\quad + \frac{1}{2}\left(\frac{\partial v^\alpha}{\partial s} + J(u^\alpha)\frac{\partial v^\alpha}{\partial t} + (\partial_\beta J)(u^\alpha)\frac{\partial u^\alpha}{\partial t}v^\beta\right)ds. \end{aligned}$$

As  $t \rightarrow \infty$ ,  $u^\alpha - u_\infty^\alpha$  is of exponential decay of weight  $\delta$ . So  $(D\bar{\partial})_{u^\alpha}$  is a compact perturbation of an operator whose restriction to the cylindrical end is

$$\frac{1}{2}\left(\frac{\partial v^\alpha}{\partial t} - J(u_\infty^\alpha)\frac{\partial v^\alpha}{\partial s}\right)dt + \frac{1}{2}\left(\frac{\partial v^\alpha}{\partial s} + J(u_\infty^\alpha)\frac{\partial v^\alpha}{\partial t}\right)ds.$$



The operator  $J(u_\infty^\alpha) \frac{\partial}{\partial s}$  is a small perturbation of  $J(0) \frac{\partial}{\partial s}$  when  $u_\infty^\alpha$  is small, which has spectrum  $\{\frac{\lambda}{m} : \lambda \in \mathbf{Z}\}$  where  $m$  is the multiplicity at  $z_0$  determined by  $\xi$ . So for any given  $\delta \in (0, \epsilon_0)$  for a sufficiently small  $\epsilon_0$ , when  $u_\infty^\alpha$  is small enough,  $\delta$  is not in the spectrum of  $J(u_\infty^\alpha) \frac{\partial}{\partial s}$ , so that the operator whose restriction to the cylindrical end is

$$\frac{1}{2} \left( \frac{\partial v^\alpha}{\partial t} - J(u_\infty^\alpha) \frac{\partial v^\alpha}{\partial s} \right) dt + \frac{1}{2} \left( \frac{\partial v^\alpha}{\partial s} + J(u_\infty^\alpha) \frac{\partial v^\alpha}{\partial t} \right) ds$$

is Fredholm, which implies that  $(D\bar{\partial})_{u^\alpha}$  is Fredholm. Hence  $(DF_0)_{\tilde{s}}$  is Fredholm for a given weight  $\delta \in (0, \epsilon_0)$  when  $\|s_\infty\|$  is small. As for the calculation of the index, we may assume that  $\tilde{s} = 0$ . Then  $(DF_0)_0$  can be perturbed to an operator whose restriction to the cylindrical end is

$$\frac{1}{2} \left( \frac{\partial v^\alpha}{\partial t} - J_0 \frac{\partial v^\alpha}{\partial s} \right) dt + \frac{1}{2} \left( \frac{\partial v^\alpha}{\partial s} + J_0 \frac{\partial v^\alpha}{\partial t} \right) ds$$

where  $J_0$  is the standard complex structure on  $\mathbf{C}^n$ , while for  $(DF)_0$  the corresponding perturbed operator is of the form of the standard  $\bar{\partial}$  on a disc neighborhood of  $z_0$ . So they have identical kernel and cokernel, hence the same index.

As for the zero sets,  $F_0^{-1}(0) \subset F^{-1}(0)$  follows from the removability of isolated singularities of pseudo-holomorphic maps. Now let  $\tilde{s} \in F^{-1}(0)$ ; restricting to  $\Sigma \setminus \{z_0\}$ ,  $\tilde{s}$  can be thought as a section of  $(TX)_{\xi_0}^*$ , and  $F_0(\tilde{s}) = 0$ . We need to show that  $\tilde{s}(z_0) \in (TX)_{f(z_0)}^\xi$  and  $\tilde{s} - \tilde{s}(z_0)$  exponentially decays with a weight at least  $\delta$ , so that  $\tilde{s}$  can be regarded as in  $L_{1,\delta}^p((TX)_{\xi_0}^*; (TX)_{f(z_0)}^\xi)$ . It is obvious that  $\tilde{s}(z_0) \in (TX)_{f(z_0)}^\xi$ . On the other hand, for any non-zero pseudo-holomorphic map  $f : D \rightarrow (\mathbf{C}^n, J)$  with  $J(0) = J_0$  and  $f(0) = 0$ , we have  $f(z) = a \cdot z^k + O(|z|^{k+1})$  for some  $k > 0$  and  $a \in \mathbf{C}^n \setminus \{0\}$  (cf. [HW]), so that  $\tilde{s} - \tilde{s}(z_0)$  decays exponentially of a weight at least  $\frac{1}{m} > \delta$ . Here  $m$  is the multiplicity at  $z_0$  determined by  $\xi$ .  $\square$

With this understood, we now move on to a different issue. Let  $f : (\Sigma, \mathbf{z}) \rightarrow X$  be a pseudo-holomorphic map with a twisted boundary condition  $\xi$ , such that  $f(\Sigma)$  lies entirely in  $\Sigma X$ . Then after deleting finitely many points  $\mathbf{z}'$  including  $\mathbf{z}$ , for any point  $p$  in  $f(\Sigma \setminus \{\mathbf{z}'\})$ ,  $G_p$  is isomorphic to a fixed group  $G$ . Then the isomorphism class of compatible systems  $\xi$ , restricted to  $\Sigma \setminus \{\mathbf{z}'\}$ , defines an isomorphism class of fiber bundles over  $\Sigma \setminus \{\mathbf{z}'\}$  with fiber  $G$ . The global sections, which extend over the whole  $\Sigma$ , form a group isomorphic to a subgroup of  $G$ . We call this group the *isotropy group* of  $(f, \xi)$ , denoted by  $G_{(f, \xi)}$ . Obviously,  $G_{(f, \xi)}$  acts linearly on the space of sections of the pull-back orbifold bundle  $(TX)_{f, \xi}^*$  defined by  $(f, \xi)$  by pointwise multiplication. Let  $g \in G_{(f, \xi)}$ ,  $z \in \mathbf{z}$ , and for a local representation of  $\xi$ , let the group homomorphism of  $\xi$  at  $z$  be given by an element  $\xi(z) \in G_{f(z)}$ ; then within the local trivialization determined by the local representation of  $\xi$ , we have  $g(z)$  lying in the centralizer of  $\xi(z)$ . Suppose  $\sigma = (f, (\Sigma, \mathbf{z}), \xi)$  is a stable map, where  $\Sigma = \cup \pi_\nu(\Sigma_\nu)$  is a semistable curve,  $\xi = (\xi_\nu)$ , where  $\xi_\nu$  is a twisted boundary condition of  $f_\nu = f \circ \pi_\nu$  satisfying Definition 2.3.3 (3). We let  $G_\nu$  be the isotropy group of  $(f_\nu, \xi_\nu)$ , and define

$$G_\sigma = \{(g_\nu) \in \prod_\nu G_\nu \mid g_\nu(z_\nu) = g_\omega(z_\omega), \text{ if } \pi_\nu(z_\nu) = \pi_\omega(z_\omega)\},$$

and call it the isotropy group of  $\sigma$ . The automorphism group  $Aut(\sigma)$  of  $\sigma$  acts on  $G_\sigma$  as automorphisms via pull-backs.

Next we have a digression. Recall that a stable curve  $x = (\Sigma, \mathbf{z})$  has a neighborhood in the Deligne-Mumford compactification which is described as

$$\frac{V_x \times B_r(\oplus_z T_{z_\nu} \Sigma_\nu \otimes T_{z_\omega} \Sigma_\omega)}{Aut(\Sigma, \mathbf{z})},$$

where  $V_x$  parametrizes the deformation of complex structures on  $x$ , and  $B_r(\oplus_z T_{z_\nu} \Sigma_\nu \otimes T_{z_\omega} \Sigma_\omega)$  gives the parameters of resolving the singularities. We denote  $V_x$  by  $V_{\text{deform}}$  and  $B_r(\oplus_z T_{z_\nu} \Sigma_\nu \otimes T_{z_\omega} \Sigma_\omega)$  by  $V_{\text{resolv}}$ . Given  $(y, \varsigma) \in V_{\text{deform}} \times V_{\text{resolv}}$ , where  $\varsigma = (\sigma_z) \in \oplus_z T_{z_\nu} \Sigma_\nu \otimes T_{z_\omega} \Sigma_\omega$ , there is an associated stable curve  $(\Sigma_{y, \varsigma}, \mathbf{z}_{y, \varsigma})$  obtained as follows. Each component  $\Sigma_\nu$  of  $\Sigma_y$  is given a Kähler metric  $\mu_y$  which is flat in a neighborhood of the singular points. This gives a Hermitian metric on each  $T_{z_\nu} \Sigma_\nu$ . For each non-zero  $\sigma_z \in T_{z_\nu} \Sigma_\nu \otimes T_{z_\omega} \Sigma_\omega$ , there is a biholomorphic map  $\Psi_{\sigma_z} : T_{z_\nu} \Sigma_\nu \setminus \{0\} \rightarrow T_{z_\omega} \Sigma_\omega \setminus \{0\}$  defined by  $u \otimes \Psi_{\sigma_z}(u) = \sigma_z$ . Let  $|\sigma_z| = R^{-2}$ , then for sufficiently large  $R$ , the map  $\exp_{z_\omega}^{-1} \circ \Psi_{\sigma_z} \circ \exp_{z_\nu}$  is a biholomorphism between  $D_{z_\nu}(R^{-1/2}) \setminus D_{z_\nu}(R^{-3/2})$  and  $D_{z_\omega}(R^{-1/2}) \setminus D_{z_\omega}(R^{-3/2})$  where  $D_{z_\nu}(R^{-1/2})$  is a disc neighborhood of  $z_\nu$  in  $\Sigma_\nu$  of radius  $(R^{-1/2})$  which is flat assuming  $R$  is sufficiently large. We glue  $\Sigma_\nu$  and  $\Sigma_\omega$  by this biholomorphism. If  $\sigma_z = 0$ , we do not make any change. Thus we obtain  $(\Sigma_{y, \varsigma}, \mathbf{z}_{y, \varsigma})$ . If we view a punctured Riemann surface as a Riemann surface with cylindrical ends, then the gluing in the above construction is equivalent to gluing  $(\frac{1}{2} \ln R, \frac{3}{2} \ln R) \times S^1$  to itself by  $t \rightarrow 2 \ln R - t$  and  $s \rightarrow -(s + \alpha)$  when resolving  $z$  with parameter  $\sigma_z = R^{-2} e^{i\alpha}$ . This construction naturally gives a Kähler metric  $\mu_{y, \varsigma}$  on  $\Sigma_{y, \varsigma}$  which coincides with the Kähler metric  $\mu_y$  on  $\Sigma_y$  outside a neighborhood of the cylindrical region. Each  $\gamma \in \text{Aut}(\Sigma, \mathbf{z})$  takes  $(\Sigma_y, \mathbf{z}_y)$  to  $(\Sigma_{\gamma(y)}, \mathbf{z}_{\gamma(y)})$  isometrically, so it acts on  $\oplus_z T_{z_\nu} \Sigma_\nu \otimes T_{z_\omega} \Sigma_\omega$ .  $\gamma$  induces an isomorphism between  $(\Sigma_{y, \varsigma}, \mathbf{z}_{y, \varsigma})$  and  $(\Sigma_{\gamma(y, \varsigma)}, \mathbf{z}_{\gamma(y, \varsigma)})$ , which is also an isometry.

Now let  $\sigma = (f, (\Sigma, \mathbf{z}), \xi)$  be the given stable map. We will put a minimal number of additional marked points on each unstable component of  $(\Sigma, \mathbf{z})$  so that it becomes stable, and the images of these additional marked points under  $f$  are invariant under  $\text{Aut}(\sigma)$ . We denote the resulting stable curve by  $(\Sigma, \mathbf{z})^+$ . Then  $(\Sigma, \mathbf{z})^+$  has a neighborhood described by  $V_{\text{deform}} \times V_{\text{resolv}} / \text{Aut}((\Sigma, \mathbf{z})^+)$ . Note that in the same way the automorphism group  $\text{Aut}(\Sigma, \mathbf{z})$  of  $(\Sigma, \mathbf{z})$ , which may be of positive dimension, also acts on  $V_{\text{deform}} \times V_{\text{resolv}}$ . If  $(y, \varsigma)' = \gamma(y, \varsigma)$  for some  $\gamma \in \text{Aut}(\Sigma, \mathbf{z})$ , then  $(y, \varsigma)'$  and  $(y, \varsigma)$  represent the same equivalence class of stable curve, if and only if  $\gamma \in \text{Aut}((\Sigma, \mathbf{z})^+)$ . If we drop the newly added marked points, then  $(y, \varsigma)'$  and  $(y, \varsigma)$  represent the same equivalence class of stable curves if and only if there is a  $\gamma \in \text{Aut}(\Sigma, \mathbf{z})$  such that  $(y, \varsigma)' = \gamma(y, \varsigma)$ . In particular, as a subgroup of  $\text{Aut}(\Sigma, \mathbf{z})$ , the automorphism group of  $\sigma$ ,  $\text{Aut}(\sigma)$ , acts on the space  $V_{\text{deform}} \times V_{\text{resolv}}$ .

Given a  $(y, \varsigma) \in V_{\text{deform}} \times V_{\text{resolv}}$ , we will construct an “almost” pseudo-holomorphic map with a canonical twisted boundary condition induced from  $\xi$ ,  $f_{y, \varsigma} : (\Sigma_{y, \varsigma}, \mathbf{z}_{y, \varsigma}) \rightarrow X$ , such that  $f_{y, \varsigma} = f_{\gamma(y, \varsigma)} \circ \gamma$  for any  $\gamma \in \text{Aut}(\sigma)$ . This is done as follows. Let  $\beta : \mathbf{R} \rightarrow [0, \infty)$  be a cut-off function such that  $\beta(t) = 1$  for  $t \leq 0$  and  $\beta(t) = 0$  for  $t \geq \frac{1}{2}$  such that  $|\beta'| \leq 4$ . Let  $\varsigma = (\sigma_z) \in V_{\text{resolv}}$  such that  $|\sigma_z| = R_z^{-2}$  is sufficiently small for all singular point  $z$ . Let  $z = \pi_\nu(z_\nu) = \pi_\omega(z_\omega)$ ,  $p_z = f(z)$ , and  $(V_{p_z}, G_{p_z}, \pi_{p_z})$  be a geodesic chart at  $p_z$ . Let the group homomorphism of  $\xi_\nu$  at  $z_\nu$  be given by  $e^{2\pi i/n_\nu} \rightarrow g_\nu$  and the group homomorphism of  $\xi_\omega$  at  $z_\omega$  be given by  $e^{2\pi i/n_\omega} \rightarrow g_\omega$ ; then  $n_\nu = n_\omega = n_z$  and  $g_\nu = g_\omega^{-1} = g_z$  in  $G_{p_z}$ . For large enough  $R_z$ , both  $f_\nu(\frac{1}{2} \ln R_z, \infty)$  and  $f_\omega(\frac{1}{2} \ln R_z, \infty)$  lie in  $\pi_{p_z}(V_{p_z})$ . We define  $f_{y, \varsigma}$  to be identical to  $f$  outside the neck region on  $\Sigma_{y, \varsigma}$ , and to equal  $\beta(t - \frac{1}{2} \ln R_z) f_\nu$  and  $\beta(t - \frac{1}{2} \ln R_z) f_\omega$  on the neck region. Note that on the part of the gluing region  $[\frac{1}{2}(\ln R_z + 1), \frac{1}{2}(3 \ln R_z - 1)] \times S^1$ ,  $f_{y, \varsigma} \equiv p_z$ .  $f_{y, \varsigma}$  inherits a twisted boundary condition  $\xi_{y, \varsigma}$  from  $\xi$  canonically, whose representative on the neck region can be obtained by gluing together  $(f_\nu, \xi_\nu)$  and  $(f_\omega, \xi_\omega)$  suitably. Since the images of  $f_\nu$  and  $f_\omega$  lie in fixed point sets which have a cone structure, so  $\beta(t - \frac{1}{2} \ln R_z) f_\nu$  and  $\beta(t - \frac{1}{2} \ln R_z) f_\omega$  lie in the same stratum of the canonical stratification  $X = X_{\text{reg}} \cup \widetilde{\Sigma X}_{\text{gen}}$  as  $f_\nu$  and  $f_\omega$  do respectively, from which it follows that the isotropy group  $G_\sigma$  of  $\sigma$  is naturally isomorphic to the isotropy group  $G_{y, \varsigma}$  of each  $(f_{y, \varsigma}, (\Sigma_{y, \varsigma}, \mathbf{z}_{y, \varsigma}), \xi_{y, \varsigma})$ , which is compatible with the action of  $\text{Aut}(\sigma)$ . Finally, it is easy to see that  $(f_{y, \varsigma}, \xi_{y, \varsigma}) = (f_{\gamma(y, \varsigma)}, \xi_{\gamma(y, \varsigma)}) \circ \gamma$  for any  $\gamma \in \text{Aut}(\sigma)$ .

We define for  $p > 2$

$$L_{1, \delta}^p((TX)_\sigma^*; (TX)^\xi) = \{(u_\nu) \in \oplus_\nu L_{1, \delta}^p((TX)_{\xi_\nu}^*; (TX)_{f_\nu(z_\nu)}^{\xi_\nu}) | u_{\nu, \infty} = u_{\omega, \infty}, \text{ if } \pi_\nu(z_\nu) = \pi_\omega(z_\omega)\},$$

and

$$L_{\delta}^p((TX)_{\sigma}^* \otimes \Lambda^{0,1}) = \{(u_{\nu}) \in \oplus_{\nu} L_{\delta}^p((TX)_{\xi_{\nu}}^* \otimes \Lambda^{0,1}(\Sigma_{\nu} \setminus \{z_{\nu}\}))\},$$

and we define  $L_{1,\delta}^p((TX)_{y,\varsigma}^*; (TX)^{\xi_{y,\varsigma}})$  and  $L_{\delta}^p((TX)_{y,\varsigma}^* \otimes \Lambda^{0,1})$  similarly, but with the understanding that the norm at each neck region at  $z$  is defined with a weight function  $e^{\delta\tau}$ , where on the neck  $(\frac{1}{2} \ln R_z, \frac{3}{2} \ln R_z)$  the function  $\tau$  is given by

$$(3.2.1) \quad \tau(t) = \begin{cases} t & t \in (\frac{1}{2} \ln R_z, \ln R_z - \frac{1}{2}) \\ 2 \ln R_z - t & t \in (\ln R_z + \frac{1}{2}, \frac{3}{2} \ln R_z) \\ \ln R_z - \frac{1}{2} & t \in [\ln R_z - \frac{1}{2}, \ln R_z + \frac{1}{2}]. \end{cases}$$

The group  $G_{\sigma}$  acts linearly on  $L_{1,\delta}^p((TX)_{\sigma}^*; (TX)^{\xi})$  and  $L_{\delta}^p((TX)_{\sigma}^* \otimes \Lambda^{0,1})$ , and  $G_{y,\varsigma}$  acts linearly on  $L_{1,\delta}^p((TX)_{y,\varsigma}^*; (TX)^{\xi_{y,\varsigma}})$  and  $L_{\delta}^p((TX)_{y,\varsigma}^* \otimes \Lambda^{0,1})$ . Moreover, the group  $Aut(\sigma)$  acts linearly on  $L_{1,\delta}^p((TX)_{\sigma}^*; (TX)^{\xi})$  and  $L_{\delta}^p((TX)_{\sigma}^* \otimes \Lambda^{0,1})$ , covering the action of  $Aut(\sigma)$  on  $(\Sigma, \mathbf{z})$ . If we let  $g \rightarrow \gamma^*(g)$  be the automorphism on  $G_{\sigma}$  induced by pull-back via  $\gamma \in Aut(\sigma)$ , then for any section  $u$  in  $L_{1,\delta}^p((TX)_{\sigma}^*; (TX)^{\xi})$  or  $L_{\delta}^p((TX)_{\sigma}^* \otimes \Lambda^{0,1})$ , we have

$$(\gamma_*)^{-1} \circ g \circ \gamma_*(u) = \gamma^*(g)(u).$$

We define  $\Gamma_{\sigma}$  to be the group generated by  $G_{\sigma}$  and  $Aut(\sigma)$  with the above relation; then  $\Gamma_{\sigma}$  is a finite group and the short sequence

$$(3.2.2) \quad 1 \rightarrow G_{\sigma} \rightarrow \Gamma_{\sigma} \rightarrow Aut(\sigma) \rightarrow 1$$

is exact. By definition,  $\Gamma_{\sigma}$  acts on  $L_{1,\delta}^p((TX)_{\sigma}^*; (TX)^{\xi})$  and  $L_{\delta}^p((TX)_{\sigma}^* \otimes \Lambda^{0,1})$  linearly. The group  $\Gamma_{\sigma}$  also acts on  $V_{deform} \times V_{resolv}$  if we let the action of  $G_{\sigma}$  be trivial. Finally, the maps  $f_{y,\varsigma}$  are “almost” pseudo-holomorphic in the sense that

$$\|\bar{\partial}_{\Sigma_{y,\varsigma}} \tilde{f}_{y,\varsigma}\|_{L_{\delta}^p(\Sigma_{y,\varsigma})} \leq C \left( \sum_z e^{-1/2(m_z^{-1}-\delta) \ln R_z} + \|y\| \right)$$

where  $z$  runs over all the singular points on  $\Sigma$ ,  $m_z$  is the multiplicity at  $z$ , and  $\varsigma = (\sigma_z)$  with  $|\sigma_z| = R_z^{-2}$ .

Thus we obtain a family of nonlinear Fredholm maps

$$F_{\sigma} : L_{1,\delta}^p((TX)_{\sigma}^*; (TX)^{\xi}) \rightarrow L_{\delta}^p((TX)_{\sigma}^* \otimes \Lambda^{0,1}) \quad \text{and} \quad F_{y,\varsigma} : L_{1,\delta}^p((TX)_{y,\varsigma}^*; (TX)^{\xi_{y,\varsigma}}) \rightarrow L_{\delta}^p((TX)_{y,\varsigma}^* \otimes \Lambda^{0,1})$$

which are the nonlinear Cauchy-Riemann equations associated to  $\sigma$  and the “almost” pseudo-holomorphic maps  $f_{y,\varsigma}$  with twisted boundary conditions  $\xi_{y,\varsigma}$ , parametrized by  $V_{deform} \times V_{resolv}$ . We will construct a Kuranishi model for this family.

First we show that  $F_{\sigma}$  and  $F_{y,\varsigma}$  have the same index.

**Lemma 3.2.4:** *Let  $\sigma$  be in  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ . Write  $\iota(\mathbf{x}) = \sum_{i=1}^k \iota_{(g_i)}$  where  $\mathbf{x} = (X_{(g_1)}, \dots, X_{(g_k)})$  and  $\iota_{(g)}$  is the degree shifting number for  $X_{(g)}$  (cf. [CR1]). The nonlinear Fredholm maps  $F_{\sigma}$  and  $F_{y,\varsigma}$  have the same index, which equals*

$$2c_1(TX) \cdot f_*([\Sigma]) + 2n(1 - g_{\Sigma}) - 2\iota(\mathbf{x}).$$

**Proof:** Without loss of generality, we may assume that  $\Sigma_{y,\varsigma}$  is obtained from  $\Sigma$  by resolving the only singular point  $z$ , and assume that  $\pi_{\nu}(z_{\nu}) = \pi_{\omega}(z_{\omega}) = z$ . Recall that, for a pseudo-holomorphic

map  $f$  with a twisted boundary condition  $\xi_0$  from a smooth Riemann surface  $\Sigma_0$ , the index of the nonlinear Cauchy-Riemann equation  $F$  is given by

$$2c_1((TX)_{\xi_0}^*)([\Sigma_0]) + 2n(1 - g_{\Sigma_0}),$$

which equals

$$2c_1(TX) \cdot f_*([\Sigma_0]) - 2 \sum_{i=1}^k \sum_{j=1}^n \frac{m_{i,j}}{m_i} + 2n(1 - g_{\Sigma_0})$$

where  $m_i$  is the multiplicity of the orbifold point  $z_i$ , and  $m_{i,j}$  is obtained from the representation of the group homomorphism of  $\xi_0$  at  $z_i$ . (cf. Proposition 4.1.4 in [CR1].) Hence we have

$$\begin{aligned} \text{index} F_\sigma &= \sum_\nu \text{index} F_{\Sigma_\nu} - \dim(TX)_{f(z)}^\xi \\ &= \sum_\nu 2c_1(TX) \cdot (f_\nu)_*([\Sigma_\nu]) - 2 \sum_{i=1}^k \sum_{j=1}^n \frac{m_{i,j}}{m_i} - 2 \sum_{j=1}^n \frac{(m_{z_\nu,j} + m_{z_\omega,j})}{m_z} \\ &\quad + \sum_\nu 2n(1 - g_{\Sigma_\nu}) - \dim(TX)_{f(z)}^\xi \\ &= 2c_1(TX) \cdot f_*([\Sigma]) - 2 \sum_{i=1}^k \sum_{j=1}^n \frac{m_{i,j}}{m_i} + 2n(1 - g_\Sigma) \end{aligned}$$

where  $i = 1, \dots, k$  runs over the index of the marked points  $\mathbf{z}$ . Here we used the fact that

$$2 \sum_{j=1}^n \frac{(m_{z_\nu,j} + m_{z_\omega,j})}{m_z} + \dim(TX)_{f(z)}^\xi = 2n.$$

The lemma follows from the fact that  $\iota(\mathbf{x}) = \sum_{i=1}^k \sum_{j=1}^n \frac{m_{i,j}}{m_i}$ . □

Let  $E_\sigma$  be a subspace of  $L_\delta^p((TX)_\sigma^* \otimes \Lambda^{0,1})$  which satisfies the following conditions:

- Let  $L_\sigma$  be the linearization of  $F_\sigma$  at 0; then  $L_\delta^p((TX)_\sigma^* \otimes \Lambda^{0,1})$  is spanned by  $E_\sigma$  and the image of  $L_\sigma$ , which is a closed, finite codimensional subspace of  $L_\delta^p((TX)_\sigma^* \otimes \Lambda^{0,1})$ .
- $E_\sigma$  is a finite dimensional effective representation of  $\Gamma_\sigma$ , inherited from the action of  $\Gamma_\sigma$  on  $L_\delta^p((TX)_\sigma^* \otimes \Lambda^{0,1})$ .
- $E_\sigma$  consists of  $C^\infty$  sections, and there exists a sufficiently large number  $R_\sigma > 0$  such that the support of each section in  $E_\sigma$  is contained in  $t \leq R_\sigma$  on the cylindrical ends.

Such a subspace  $E_\sigma$  certainly exists.

The main result in the subsection on the construction of a Kuranishi neighborhood is summarized in the following

**Proposition 3.2.5:** *Let  $V_\sigma^+$  be the finite dimensional subspace in  $L_{1,\delta}^p((TX)_\sigma^*; (TX)^\xi)$  defined by  $V_\sigma^+ = L_\sigma^{-1}(E_\sigma)$ . Suppose that  $V_{\text{deform}} \times V_{\text{resolv}}$  consists of sufficiently small  $(y, \varsigma)$ ; in particular if  $\varsigma = (\sigma_z)$ , we require that  $|\sigma_z| = R_z^{-2} < R_\sigma^{-2}$ . We take an isomorphism  $\theta_{y,\varsigma} : \Lambda^1(\Sigma) \rightarrow \Lambda^1(\Sigma)$  so that  $\theta_{y,\varsigma}$  induces an isomorphism  $\Lambda^{0,1}(\Sigma)|_{\text{supp} E_\sigma} \rightarrow \Lambda^{0,1}(\Sigma_{y,\varsigma})|_{\text{supp} E_\sigma}$  which is compatible with the action of  $\text{Aut}(\sigma)$ . Now  $E_{y,\varsigma} = \theta_{y,\varsigma}(E_\sigma)$  can be regarded as a subspace of  $L_\delta^p((TX)_{y,\varsigma}^* \otimes \Lambda^{0,1})$ . Let*

$$Z_r = \{u | F_\sigma(u) \in E_\sigma, \|u\| \leq r\} \cup_{(y,\varsigma) \in V_{\text{deform}} \times V_{\text{resolv}}} \{u | F_{y,\varsigma}(u) \in E_{y,\varsigma}, \|u\| \leq r\}.$$

Then for sufficiently small  $r > 0$ , there is a  $\Gamma_\sigma$ -invariant open neighborhood  $U_r^+$  in  $V_{\text{deform}} \times V_{\text{resolv}} \times V_\sigma^+$ , and a  $\Gamma_\sigma$ -equivariant one to one and onto map  $\psi_r : U_r^+ \rightarrow Z_r$ , which is a diffeomorphism when restricted to the “slice” over each  $(y, \varsigma)$  or  $\sigma$ . Let  $s_r : U_r^+/\Gamma_\sigma \rightarrow E_\sigma/\Gamma_\sigma$  be the composition of  $\psi_r$  with  $F_\sigma$  and  $(\theta_{y,\varsigma})^{-1} \circ F_{y,\varsigma}$ . Then  $s_r$  is continuous. Finally, suppose each newly added marked point  $z$  on  $\Sigma$  is chosen such that  $f(z)$  and  $f(z')$  have the same orbit type for nearby points  $z'$  and  $f$  is an embedding at  $z$ . For each  $z$ , choose a  $G_{f(z)}$ -invariant subspace  $W_z$  in  $TX_{f(z)}$  of codimension two such that  $W_z$  is orthogonal to  $\text{Im } df(z)$  and  $\sqcup_z W_z$  is  $\Gamma_\sigma$ -invariant. We define

$$V_\sigma = \{u \in V_\sigma^+ | u(z) \in W_z, \text{ for all newly added marked points } z\}.$$

Then  $V_\sigma$  is a  $\Gamma_\sigma$ -invariant subspace of  $V_\sigma^+$ . Let  $U_r$  be the intersection of  $U_r^+$  with  $V_{\text{deform}} \times V_{\text{resolv}} \times V_\sigma$ ; then  $\psi_r$  restricted to  $U_r/\Gamma_\sigma$  induces a homeomorphism between  $s_r^{-1}(0) \in U_r/\Gamma_\sigma$  and a neighborhood of  $\sigma$  in the moduli space  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ . The dimension of this Kuranishi neighborhood is

$$2c_1(TX) \cdot A + 2(n-3)(1-g) + 2k - 2\iota(\mathbf{x}).$$

Here  $\iota(\mathbf{x})$  is defined to be  $\sum_{i=1}^k \iota(g_i)$  for  $\mathbf{x} = (X_{(g_1)}, \dots, X_{(g_k)})$ , where  $\iota(g)$  is the degree shifting number for  $X_{(g)}$  (cf. [CR1]).

**Proof:** There are three technical issues in the proof. The first one is to show that the Kuranishi model of each individual map  $F_\sigma$  or  $F_{y,\varsigma}$  has a uniform size. The second one is to show that  $\psi_r$  is surjective to a neighborhood of  $\sigma$  in  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ . The point here is that the topology of  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  is defined by the weak  $C^\infty$  topology plus the  $C^0$  topology on the neck region, while the topology of  $U_r^+$  is given by weighted Sobolev norms. A delicate estimate for pseudo-holomorphic maps is needed here. The third one is to prove that when restricted to  $s_r^{-1}(0) \cap U_r/\Gamma_\sigma$ , the map  $\psi_r$  is a homeomorphism onto a neighborhood of  $\sigma$  in  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ , which involves choosing representatives of stable maps.

We choose a  $\Gamma_\sigma$ -invariant decomposition  $L_{1,\delta}^p((TX)_\sigma^*; (TX)^\xi) = V_\sigma^+ \oplus V_\sigma'$  and a  $\Gamma_\sigma$ -invariant decomposition  $L_\delta^p((TX)_\sigma^* \otimes \Lambda^{0,1}) = E'_\sigma \oplus E_\sigma$  with a decomposition ratio  $c$  such that  $L_\sigma : V_\sigma' \rightarrow E'_\sigma$  is an isomorphism. Let  $(y, \varsigma)$  be in  $V_{\text{deform}} \times V_{\text{resolv}}$ , where  $\varsigma = (\sigma_z)$  with  $|\sigma_z| = R_z^{-2}$ . For each  $R_z$ , we define a cut-off function  $\rho_{R_z}$  such that  $\rho_{R_z} \equiv 1$  for  $t \leq \ln R_z$  and  $\rho_{R_z} \equiv 0$  for  $t \geq \frac{1}{2}(3 \ln R_z - 1)$ , and  $|\rho'_{R_z}| \leq 4/\ln R_z$ . With  $\rho_{R_z}$ , for each section  $u$  in  $L_{1,\delta}^p((TX)_\sigma^*; (TX)^\xi)$ , we can construct a section  $\#u$  in  $L_{1,\delta}^p((TX)_{y,\varsigma}^*; (TX)^{\xi_{y,\varsigma}})$  by  $\#u = \rho_{R_z} u_{z_\nu} + (1 - \rho_{R_z}) u_{z_\omega}$  on the neck region at  $z$ . The map  $\# : u \rightarrow \#u$  is  $\Gamma_\sigma$ -equivariant. Now given a section  $x$  in  $L_\delta^p((TX)_{y,\varsigma}^* \otimes \Lambda^{0,1})$ , we can think of  $(\theta_{y,\varsigma})^{-1}(x)$  as a section in  $L_\delta^p((TX)_\sigma^* \otimes \Lambda^{0,1})$  which equals zero on  $t \geq \ln R_z$ . Then  $(\theta_{y,\varsigma})^{-1}(x)$  admits a decomposition as  $(\theta_{y,\varsigma})^{-1}(x) = L_\sigma u_0 + (\theta_{y,\varsigma})^{-1}(e_0)$  where  $L_\sigma u_0 \in E'_\sigma$  and  $e_0 \in E_{y,\varsigma}$ , and  $u_0 \in V_\sigma'$ . Let  $L_{y,\varsigma}$  be the linearization of  $F_{y,\varsigma}$  at 0, and consider  $L_{y,\varsigma}(\#u_0)$ . Let  $x_1$  be the difference  $\theta_{y,\varsigma}(L_\sigma u_0) - L_{y,\varsigma}(\#u_0)$ . Then we have  $x = L_{y,\varsigma}(\#u_0) + e_0 + x_1$ , and  $x_1$  satisfies the following estimate:

$$\begin{aligned} \|x_1\|_{L_\delta^p} &\leq C(\|y\| + \sum_z (\ln R_z)^{-1}) \|u_0\|_{L_{1,\delta}^p} \\ &\leq C(\|y\| + \sum_z (\ln R_z)^{-1}) \|L_\sigma^{-1}\| \|L_\sigma u_0\|_{L_\delta^p} \\ &\leq C(\|y\| + \sum_z (\ln R_z)^{-1}) \|L_\sigma^{-1}\| \|c\| \|x\|_{L_\delta^p}. \end{aligned}$$

So when  $(\|y\| + \sum_z (\ln R_z)^{-1})$  is sufficiently small, we have  $\|x_1\| \leq \frac{1}{2}\|x\|$ . We can continue this procedure and get  $x_1 = L_{y,\varsigma}(\#u_1) + e_1 + x_2$  and  $\|x_2\| \leq \frac{1}{2}\|x_1\|$ ,  $\dots$ ,  $x_n = L_{y,\varsigma}(\#u_n) + e_n + x_{n+1}$

and  $\|x_{n+1}\| \leq \frac{1}{2}\|x_n\|, \dots$ . Let  $\eta_1(x) = \#(\sum_{n=0}^{\infty} u_n)$  and  $\eta_2(x) = \sum_{n=0}^{\infty} e_n$ . Then we have

$$\begin{aligned} \|\eta_1(x)\|_{L_{1,\delta}^p} &\leq C \sum_{n=0}^{\infty} \|u_n\|_{L_{1,\delta}^p} \leq C \sum_{n=0}^{\infty} \|L_{\sigma}^{-1}\| \|L_{\sigma}(u_n)\|_{L_{\delta}^p} \\ &\leq C \sum_{n=0}^{\infty} \|L_{\sigma}^{-1}\| c \|x_n\|_{L_{\delta}^p} \leq 3Cc \|L_{\sigma}^{-1}\| \|x\|_{L_{\delta}^p} \end{aligned}$$

and

$$\|\eta_2(x)\|_{L_{\delta}^p} \leq \sum_{n=0}^{\infty} \|e_n\|_{L_{\delta}^p} \leq \sum_{n=0}^{\infty} c \|x_n\|_{L_{\delta}^p} \leq 3c \|x\|_{L_{\delta}^p},$$

and

$$x = L_{y,\varsigma}(\eta_1(x)) + \eta_2(x).$$

It is clear that  $E_{y,\varsigma} \subset \text{kernel}(\eta_1)$  so that if we let  $E'_{y,\varsigma} = \text{Im}(L_{y,\varsigma} \circ \eta_1)$ , we obtain a decomposition of  $L_{\delta}^p((TX)_{y,\varsigma}^* \otimes \Lambda^{0,1}) = E'_{y,\varsigma} \oplus E_{y,\varsigma}$ , which is  $\Gamma_{\sigma}$ -equivariant and of uniformly bounded ratio. Let  $V'_{y,\varsigma} = \text{Im}(\eta_1)$ ,  $V_{y,\varsigma}^+ = L_{y,\varsigma}^{-1}(E_{y,\varsigma})$ ; then we have a  $\Gamma_{\sigma}$ -equivariant decomposition of  $L_{1,\delta}^p((TX)_{y,\varsigma}^*; (TX)^{\xi_{y,\varsigma}}) = V_{y,\varsigma}^+ \oplus V'_{y,\varsigma}$ , and  $\eta_1 = L_{y,\varsigma}^{-1} : E'_{y,\varsigma} \rightarrow V'_{y,\varsigma}$  with uniformly bounded norm. The decomposition

$$L_{1,\delta}^p((TX)_{y,\varsigma}^*; (TX)^{\xi_{y,\varsigma}}) = V_{y,\varsigma}^+ \oplus V'_{y,\varsigma}$$

also has a uniformly bounded ratio because it is given by

$$u = (u - \eta_1(L_{y,\varsigma}(u))) + \eta_1(L_{y,\varsigma}(u)),$$

and  $\|\eta_1(L_{y,\varsigma}(u))\| \leq C\|u\|$ . Finally, we can easily verify that  $\|D^2F_{y,\varsigma}(0)\|$  is uniformly bounded.

So basically we have shown that this family of Kuranishi models has a uniform size. In order to construct a Kuranishi neighborhood for this family of maps, we only need to construct a family of isomorphisms  $\eta_{y,\varsigma} : V_{\sigma}^+ \rightarrow V_{y,\varsigma}^+$  with uniformly bounded norms. For  $u \in V_{\sigma}^+$ , we define  $\eta_{y,\varsigma}(u) = \#u - \eta_1(L_{y,\varsigma}(\#u))$ . We need to show that for sufficiently small  $(y, \varsigma)$ , there is a constant  $C$  such that  $\|\eta_{y,\varsigma}(u)\| \geq C\|u\|$ . Suppose that this is not the case. Then there is a sequence of  $(y, \varsigma)_n$  going to  $\sigma$ , a sequence of  $u_n$  with  $\|u_n\| = 1$  and  $\eta_{(y,\varsigma)_n}(u_n) \rightarrow 0$ . But this contradicts the fact that  $\eta_1(\theta_{(y,\varsigma)_n}(L_{\sigma}u_n)) = 0$  and  $\|\theta_{(y,\varsigma)_n}(L_{\sigma}(u_n)) - L_{(y,\varsigma)_n}(\#u_n)\| = o(\|u_n\|) \rightarrow 0$ . Hence we have constructed the Kuranishi model of this family of maps. The map  $s_r : U_r^+/\Gamma_{\sigma} \rightarrow E_{\sigma}/\Gamma_{\sigma}$  is obviously continuous.

Next we will show that if there is a sequence  $\sigma_n$  in  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  that is convergent to  $\sigma$ , which is represented by stable maps  $(f_n, (\Sigma_n, \mathbf{z}_n), \xi_n)$  such that after adding extra marked points,  $(f_n^+, (\Sigma_n, \mathbf{z}_n)^+, \xi_n^+)$  converges to  $(f^+, (\Sigma, \mathbf{z})^+, \xi^+)$ , then for large enough  $n$ ,  $(f_n, (\Sigma_n, \mathbf{z}_n), \xi_n)$  lies in the image of  $\psi_r$ . It is easily seen that we may assume that  $(f^+, (\Sigma, \mathbf{z})^+, \xi^+)$  is the stable map after we add the minimal number of marked points of our choice. For large  $n$ , suppose  $(\Sigma_n, \mathbf{z}_n)^+ = (\Sigma_{(y,\varsigma)_n}, \mathbf{z}_{(y,\varsigma)_n})$ ; then it is easily seen that  $\tilde{f}_n$  equals  $\text{Exp} \circ \tilde{f}_{(y,\varsigma)_n} \circ \tilde{s}_n$  for some  $\tilde{s}_n$  in  $L_{1,\delta}^p((TX)_{(y,\varsigma)_n}^*; (TX)^{\xi_{(y,\varsigma)_n}})$ . What we need to show is that for sufficiently large  $n$ ,  $\|\tilde{s}_n\|_{L_{1,\delta}^p} < r$ . For this part, we need Lemma 2.3.11. Recall that if  $(f_n^+, (\Sigma_n, \mathbf{z}_n)^+, \xi_n^+)$  converges to  $(f^+, (\Sigma, \mathbf{z})^+, \xi^+)$ , then the following holds. First, for each  $\mu > 0$ , when  $n$  is sufficiently large, the restriction of  $\tilde{f}_n^+$  to  $\Sigma_{y_n, \varsigma_n} \setminus W_n(\mu)$  converges to  $\tilde{f}^+$  in the  $C^{\infty}$  topology as a  $C^{\infty}$  map with an isomorphism class of compatible systems. Secondly,  $\lim_{\mu \rightarrow 0} \limsup_{n \rightarrow \infty} \text{Diam}(f_n(W_{z,n}(\mu))) = 0$  for each singular point  $z$  of  $\Sigma$ . Here

$$W_{z,n}(\mu) = (D_{z_{\nu}}(\mu) \setminus D_{z_{\nu}}(R_{z,n}^{-1})) \cup (D_{z_{\omega}}(\mu) \setminus D_{z_{\omega}}(R_{z,n}^{-1})), \text{ and } W_n(\mu) = \cup_z W_{z,n}(\mu).$$

We pick a  $\mu > 0$  so that for large  $n$ ,  $\text{Diam}(f_n(W_{z,n}(\mu))) < \epsilon$  where the  $\epsilon$  here is referred to Lemma 2.3.11. Then on the neck region  $[\ln \mu^{-1}, 2 \ln R_{z,n} - \ln \mu^{-1}] \times S^1$ , we have

$$\left| \frac{\partial f_n}{\partial t}(t, s) \right| + \left| \frac{\partial f_n}{\partial s}(t, s) \right| \leq C e^{-\frac{1}{m_z}(\tau(t) - \ln \mu^{-1})},$$

where  $\tau(t)$  is defined in (3.2.1). Then it follows that on the neck region

$$[2(1/m_z - \delta)^{-1} \ln \mu^{-1}, 2 \ln R_{z,n} - 2(1/m_z - \delta)^{-1} \ln \mu^{-1}] \times S^1,$$

the  $L_{1,\delta}^p$  norm of  $\tilde{s}_n$  is bounded by a term  $C e^{-\ln \mu^{-1}}$ . So we can pick  $\mu > 0$  small enough so that  $C e^{-\ln \mu^{-1}} < \frac{r}{4}$ . Then for this fixed  $\mu$ , by weak  $C^\infty$  convergence, for large enough  $n$ , the  $L_{1,\delta}^p$  norm of  $\tilde{s}_n$  on the rest is bounded by  $\frac{r}{4}$ , so that the whole  $L_{1,\delta}^p$  norm of  $\tilde{s}_n$  is bounded by  $r$ . Hence  $(f_n, (\Sigma_n, \mathbf{z}_n), \xi_n)$  lies in the image of  $\psi_r$  for large  $n$ .

Now we prove that when restricted to  $s_r^{-1}(0) \cap U_r/\Gamma_r$ , the map  $\psi_r$  is a homeomorphism onto a neighborhood of  $\sigma$  in  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ . We first clarify the issue that different sections of the same orbifold bundle may define the same  $C^\infty$  map under the exponential map  $\text{Exp}$ . Suppose there are two sections  $\tilde{s}_1$  and  $\tilde{s}_2$  such that  $\text{Exp} \circ \bar{f}_{y,\varsigma} \circ \tilde{s}_1 = \text{Exp} \circ \bar{f}_{y,\varsigma} \circ \tilde{s}_2$  gives the same pseudo-holomorphic map  $f'$ . We take a geodesic compatible system  $\{\tilde{f}_{UU'}, \lambda\}$  of  $(f, \xi)$ . Then it induces two isomorphic compatible systems  $\{\tilde{f}'_{i, UU'_i}, \lambda_i\}$  for  $i = 1, 2$  of  $f'$  through  $\tilde{s}_1$  and  $\tilde{s}_2$ . Now by the unique continuity property of pseudo-holomorphic maps, we have an isomorphism  $\delta_U$  for each  $U$  between the uniformizing system  $V'_1$  of  $U'_1$  to  $V'_2$  of  $U'_2$  such that  $\tilde{f}'_{2, UU'_2} = \delta_U \circ \tilde{f}'_{1, UU'_1}$ , and  $\delta_U$  can be chosen so that  $\lambda_2 = \delta \circ \lambda_1 \circ (\delta)^{-1}$ . Then one can check that this means that the collection  $\{\delta_U\}$  defines an element  $g$  in  $G_\sigma$ , and  $\tilde{s}_2 = g \cdot \tilde{s}_1$ . Now we prove that  $\psi_r$  is a homeomorphism onto a neighborhood of  $\sigma$  in  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  when restricted to  $s_r^{-1}(0) \cap U_r/\Gamma_\sigma$ . For the surjectivity, we observe that for a stable map  $(f', (\Sigma', \mathbf{z}'), \xi')$  sufficiently close to  $\sigma$ ,  $f'(\Sigma')$  will intersect  $\text{Exp}(W_z)$  transversally at finitely many points, say  $f'(z'_i)$ , for each newly added marked point  $z$ . We mark the one, say  $z'$ , on  $\Sigma'$  which is closest to  $z$  and obtain a stable curve  $(\Sigma', \mathbf{z}')^+$ . By what we have shown in the previous paragraph, there is a  $(y, \varsigma, \tilde{s}) \in V_{\text{deform}} \times V_{\text{resolv}} \times V_\sigma^+$  such that  $((f')^+, (\Sigma', \mathbf{z}')^+, (\xi')^+)$  is given by  $\text{Exp} \circ \bar{f}_{y,\varsigma} \circ \eta_{y,\varsigma}(\tilde{s})$ . Note that by our choice of the new marked point  $z'$ , we have  $\eta_{y,\varsigma}(\tilde{s}(z)) \in W_z$ , but  $\tilde{s}(z)$  may not lie in  $W_z$ , which is the condition for  $\tilde{s}$  to be in  $V_\sigma$ . We fix this problem in the following way. Note that  $\|\tilde{s}(z) - \eta_{y,\varsigma}(\tilde{s}(z))\| \leq C\|(y, \varsigma)\|$ . We take a section  $\tilde{s}' \in V_\sigma$  such that  $\|\tilde{s}'(z) - \tilde{s}(z)\| \leq C\|(y, \varsigma)\|$ , and consider the intersection of  $f'(\Sigma')$  with  $\text{Exp} \circ \bar{f}_{y,\varsigma} \circ \eta_{y,\varsigma}(t\tilde{s}'(z))$ , which is transversally at finitely many points when  $\|(y, \varsigma)\|$  is small enough. We replace the new marked point  $z'$  by one of these new intersection points which is closest to  $z$ , call it  $z'_1$ , then with  $z'_1$ ,  $(\Sigma', \mathbf{z}')^+$  is represented by  $(\Sigma_{(y,\varsigma)_1}, \mathbf{z}_{(y,\varsigma)_1})$  for some  $(y, \varsigma)_1$  such that  $\|(y, \varsigma)_1 - (y, \varsigma)\| \leq C\|(y, \varsigma)\|^2$ . Let  $((f')^+, (\Sigma', \mathbf{z}')^+, (\xi')^+)$  be given by  $\text{Exp} \circ \bar{f}_{(y,\varsigma)_1} \circ \eta_{(y,\varsigma)_1}(\tilde{s}_1)$ . Then  $\tilde{s}_1 = (\eta_{(y,\varsigma)_1})^{-1} \circ \eta_{y,\varsigma}(\tilde{s}')$ . The upshot is that although  $\tilde{s}_1$  may not be in  $V_\sigma$ , we can still find a section  $\tilde{s}'_1 \in V_\sigma$  such that  $\|\tilde{s}'_1 - \tilde{s}_1\| \leq C\|(y, \varsigma)\|^2$ , an improved estimate. We repeat this process and take the limit, we find a  $(y, \varsigma)_\infty$  and a section  $\tilde{s}_\infty \in V_\sigma$  such that  $((f')^+, (\Sigma', \mathbf{z}')^+, (\xi')^+)$  is given by  $\text{Exp} \circ \bar{f}_{(y,\varsigma)_\infty} \circ \eta_{(y,\varsigma)_\infty}(\tilde{s}_\infty)$ . Hence the surjectivity of  $\psi_r$ . As for the injectivity of  $\psi_r$ , we will first show that if  $(y, \varsigma, \tilde{s})$  and  $\gamma(y, \varsigma, \tilde{s})$  represent the same stable map  $(f', (\Sigma', \mathbf{z}'), \xi')$  for some  $\gamma \in \text{Aut}(\Sigma, \mathbf{z})$  which lies in a small neighborhood of identity (this corresponds to the effect of perturbing the newly added marked points by the action of  $\gamma$ ), then if both  $\tilde{s}$  and  $\gamma(\tilde{s})$  lie in  $V_\sigma$ , we must have  $\gamma = \text{id}$ . This roughly follows from the following consideration: the difference between  $\eta_{y,\varsigma}(\tilde{s}(z))$  and  $\eta_{\gamma(y,\varsigma)}(\gamma(\tilde{s})(z))$  in the direction transversal to  $W_z$  is measured by  $C\|\gamma - \text{id}\|$ . This must also be true for  $\tilde{s}(z)$  and  $\gamma(\tilde{s})(z)$  on the one hand, since  $\|\gamma(y, \varsigma) - (y, \varsigma)\| \leq C\|\gamma - \text{id}\|\|(y, \varsigma)\|$  so that the effect of  $\eta_{y,\varsigma}$  and  $\eta_{\gamma(y,\varsigma)}$  can be ignored, but on the other hand, by the assumption, both  $\tilde{s}$  and  $\gamma(\tilde{s})$  lie in  $V_\sigma$ , a contradiction. Now the injectivity of  $\psi_r$  restricted to  $s_r^{-1}(0) \cap U_r/\Gamma_\sigma$  follows

from the fact that, up to a factor in  $G_\sigma$ , if  $(y, \varsigma, \tilde{s})_1$  and  $(y, \varsigma, \tilde{s})_2$  represent the same stable map  $(f', (\Sigma', \mathbf{z}'), \xi')$ , then for some  $\gamma \in \text{Aut}(\sigma)$  and  $\gamma_0 \in \text{Aut}(\Sigma, \mathbf{z})$  which lies in a small neighborhood of the identity, we have  $(y, \varsigma, \tilde{s})_1 = \gamma_0 \circ \gamma((y, \varsigma, \tilde{s})_2)$ .

Finally, the dimension calculation of the Kuranishi neighborhood is a routine business, which follows from Lemma 3.2.4.

### 3.3 Construction of Kuranishi structure

In this subsection, we will patch up the local Kuranishi neighborhoods we constructed in the previous subsection to yield a Kuranishi structure of the moduli space  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ , which we will show to be stably complex, therefore carrying a canonical orientation. The heart of the construction is the fulfillment of the compatibility condition in the definition of Kuranishi structure. For this purpose, each local obstruction bundle  $E_\sigma$  cannot be chosen arbitrarily.

In the previous subsection, we constructed for each equivalence class of stable maps  $\sigma \in \overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  a Kuranishi neighborhood. In particular, a neighborhood of  $\sigma$  in  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  is identified with  $s_r^{-1}(0) \cap U_r / \Gamma_\sigma$  via the map  $\psi_r$  for some sufficiently small  $r > 0$ . We first cover the moduli space  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  by finitely many of this kind of open set, say  $\{O_i : i = 1, \dots, N\}$ , at  $\sigma_i \in \overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  such that there is a closed subset  $\hat{O}_i \subset O_i$  and  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  is also covered by  $\{\hat{O}_i\}$ . We assume that the Kuranishi neighborhood at  $\sigma_i$  is constructed with the choice of obstruction bundle  $E_i$  and a representative stable map  $(f_i, (\Sigma_i, \mathbf{z}_i), \xi_i)$ . We denote by  $U_i$  the Kuranishi neighborhood at  $\sigma_i$  such that  $O_i = \psi_i(s_{r_i}^{-1}(0) \cap U_i / \Gamma_i)$  where  $\Gamma_i$  stands for  $\Gamma_{\sigma_i}$  and  $s_i$  and  $\psi_i$  stands for  $s_{r_i}$  and  $\psi_{r_i}$  for some small  $r_i > 0$ . Finally, for each equivalence class of stable maps  $\tau \in \overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ , we fix a representative  $(f_\tau, (\Sigma, \mathbf{z})_\tau, \xi_\tau)$ .

**Lemma 3.3.1:** *For any equivalence class of stable maps  $\tau \in \overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ , if  $\tau \in O_i$  for some  $i$ , then for any  $(y, \varsigma, \tilde{s})$  in the Kuranishi neighborhood  $U_i$  at  $\sigma_i$  representing  $\tau$ , there is a monomorphism  $\Gamma_\tau \rightarrow \Gamma_i$  whose image is the isotropy subgroup of  $(y, \varsigma, \tilde{s})$  in  $U_i$ .*

**Proof:** The existence of a monomorphism  $\Gamma_\tau \rightarrow \Gamma_i$  follows from the existence of monomorphisms  $\text{Aut}(\tau) \rightarrow \text{Aut}(\sigma_i)$  and  $G_\tau \rightarrow G_{\sigma_i}$ . When  $(\Sigma_i, \mathbf{z}_i)$  is a stable curve (i.e. no new marked points are added to  $(\Sigma_i, \mathbf{z}_i)$ ),  $\text{Aut}((\Sigma, \mathbf{z})_\tau)$  is embedded into  $\text{Aut}(\Sigma_i, \mathbf{z}_i)$  as the isotropy subgroup of  $(y, \varsigma)$ , and when  $\tau$  is sufficiently close to  $\sigma_i$ ,  $\text{Aut}(\tau)$  as a subgroup of  $\text{Aut}((\Sigma, \mathbf{z})_\tau)$  has its image lying in  $\text{Aut}(\sigma_i)$ , which also fixes  $\tilde{s}$ . When  $(\Sigma_i, \mathbf{z}_i)$  has an unstable component so that some new marked points are added, then a biholomorphism between  $(\Sigma, \mathbf{z})_\tau$  and  $(\Sigma_{y,\varsigma}, \mathbf{z}_{y,\varsigma})$  will add corresponding new marked points on  $(\Sigma, \mathbf{z})_\tau$ , and this induces a map from  $\text{Aut}(\tau)$  into  $\text{Aut}(\Sigma_i, \mathbf{z}_i)$ , whose image, when  $\tau$  is sufficiently close to  $\sigma_i$ , lies in  $\text{Aut}(\sigma_i)$  and is a monomorphism, and it fixes  $(y, \varsigma, \tilde{s})$ . As for the monomorphism  $G_\tau \rightarrow G_{\sigma_i}$ , we consider the case when both  $\tau$  and  $\sigma_i$  have only one component for simplicity. The general case follows the same way. We recall the definition of the isotropy group  $G_\tau$  of  $\tau = (f_\tau, (\Sigma, \mathbf{z})_\tau, \xi_\tau)$ . After deleting finitely many points, the image of  $f_\tau$  has the same orbit type given by a group, say  $H_\tau$ . Then the twisted boundary condition  $\xi_\tau$  defines a fiber bundle with fiber  $H_\tau$  and  $G_\tau$  is just the group of global sections of this fiber bundle. Now when  $\|\tilde{s}\|$  is sufficiently small, all points except for finitely many in  $\text{Exp} \circ \tilde{f}_{i,(y,\varsigma)} \circ \tilde{s}$  have the same orbit type which is given by a subgroup  $H_\tau$  of the group  $H_{\sigma_i}$  defining the orbit type for  $\sigma_i$ . On the other hand, the twisted boundary condition  $\xi_\tau$  is also compatible with the twisted boundary condition for  $\sigma_i$  in the sense that the corresponding fiber bundle with fiber  $H_\tau$  is a subbundle of the fiber bundle with fiber  $H_{\sigma_i}$ . From this it follows that  $G_\tau$  is a subgroup of  $G_i$ . One can also verify that if  $g \in G_i$  fixes  $\tilde{s}$ , then  $g$  lies in  $G_\tau$ . Hence the lemma.  $\square$

Now suppose an equivalence class of stable maps  $\tau$  is in  $O_i$ , and is represented by  $(y, \varsigma, \tilde{s})$  in  $U_i \cap s_i^{-1}(0)$ . By parallel transport along geodesics  $\text{Exp} \circ \tilde{f}_{i,(y,\varsigma)} \circ t\tilde{s}(z)$ , we transport the subspace



$E_{i,(y,\varsigma)}$  of  $L_\delta^p((TX)_{i,(y,\varsigma)}^* \otimes \Lambda^{0,1})$  into  $C^\infty((TX)_{(y,\varsigma,\tilde{s})}^* \otimes \Lambda^{0,1}(\Sigma_{y,\varsigma}))$ , where  $(TX)_{(y,\varsigma,\tilde{s})}^*$  is the pull-back orbifold bundle by  $\text{Exp} \circ \bar{f}_{i,(y,\varsigma)} \circ \tilde{s}$  over  $\Sigma_{y,\varsigma}$ . We denote the resulting subspace by  $E_{i,(y,\varsigma,\tilde{s})}$ . Then we have  $\gamma^*(E_{i,\gamma(y,\varsigma,\tilde{s})}) = E_{i,(y,\varsigma,\tilde{s})}$  for any  $\gamma \in \Gamma_i$ . Let  $\theta_\tau : (\Sigma, \mathbf{z})_\tau \rightarrow (\Sigma_{y,\varsigma}, \mathbf{z}_{y,\varsigma})$  be a biholomorphism sending marked points to marked points (not including new marked points on  $\Sigma_{y,\varsigma}$ ). Then the pull-back  $\theta_\tau^*(E_{i,(y,\varsigma,\tilde{s})})$  in  $C^\infty((TX)_\tau^* \otimes \Lambda^{0,1}(\Sigma_\tau))$  does not depend on the choice of  $(y, \varsigma, \tilde{s})$  and  $\theta_\tau$ , where  $(TX)_\tau^*$  is the pull-back orbifold bundle by  $(f_\tau, (\Sigma, \mathbf{z})_\tau, \xi_\tau)$ . We denote  $\theta_\tau^*(E_{i,(y,\varsigma,\tilde{s})})$  by  $E_{\tau,i}$ .

**Lemma 3.3.2:** *We can choose each obstruction bundle  $E_i$  suitably so that for any  $\tau \in \cap_i O_i$ ,  $i = i_1, \dots, i_l$ , the subspaces  $E_{\tau,i}$  of  $C^\infty((TX)_\tau^* \otimes \Lambda^{0,1}(\Sigma_\tau))$ , for  $i = i_1, \dots, i_l$ , are linearly independent.*

**Proof:** For each  $\tau \in \cap_i O_i$ ,  $i = i_1, \dots, i_l$ , we can choose finitely many points  $w_i$  on  $\Sigma_\tau$ , away from any neck or singular point region and marked points, such that in a disc neighborhood  $D_i$  of  $w_i$ , none of the sections in each  $E_{\tau,i}$  vanishes identically. For each  $i$ , we take a cut-off function  $\beta_i$  which is zero in  $D_i$ . We multiply  $\beta_i$  to each section in  $E_{\tau,j}$  for any  $j \in \{i_1, \dots, i_l\} \setminus \{i\}$ . We transport these changes on each  $E_{\tau,i}$  back to each  $E_i$ . Then for such a choice of  $E_i$ , the lemma holds for any stable map in a neighborhood of  $\tau$  in  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ . By the compactness of  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ , we can cover  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  by finitely many such neighborhoods. Hence the lemma.  $\square$

We define  $E_\tau$  to be the span of  $E_{\tau,i}$  for those  $i$ 's such that  $\tau \in \cap_i \hat{O}_i$ . Note that here we use the closed subset  $\hat{O}_i$ . We will use  $E_\tau$  for the obstruction bundle at  $\tau$ . We first collect a few properties of  $E_\tau$ .

- Let  $L_\tau$  be the linearization of the non-linear Cauchy-Riemann operator  $\bar{\partial}_\tau$  at 0; then  $L_\delta^p((TX)_\tau^* \otimes \Lambda^{0,1}(\Sigma_\tau))$  is spanned by  $E_\tau$  and the image of  $L_\tau$ , which is a closed, finite codimensional subspace of  $L_\delta^p((TX)_\tau^* \otimes \Lambda^{0,1}(\Sigma_\tau))$ .
- $E_\tau$  is a finite dimensional effective representation of  $\Gamma_\tau$ , inherited from the action of  $\Gamma_\tau$  on  $L_\delta^p((TX)_\tau^* \otimes \Lambda^{0,1}(\Sigma_\tau))$ .
- $E_\tau$  consists of  $C^\infty$  sections, and there exists a sufficiently large number  $R_\tau > 0$  such that the support of each section in  $E_\tau$  is contained in  $t \leq R_\tau$  on the cylindrical ends.

Next we introduce the space of equivalence classes of “perturbed” stable maps. Consider a  $C^\infty$  map  $\tilde{h} : (\Sigma, \mathbf{z}) \rightarrow X$  with a twisted boundary condition  $\xi$ , where  $(\Sigma, \mathbf{z})$  is a semistable curve, and  $\xi = (\xi_\nu)$  is a collection of twisted boundary conditions satisfying similar compatibility conditions as in the definition of stable maps. We also require that if  $\tilde{h}$  is constant on a component  $\Sigma_\nu$ , then  $\Sigma_\nu$  has at least three special points (marked or singular). We further assume that  $h$  is pseudo-holomorphic in a neighborhood of each singular point on  $\Sigma$ . The equivalence relation is defined as follows. Suppose  $(\tilde{h}, (\Sigma, \mathbf{z}), \xi)$  and  $(\tilde{h}', (\Sigma', \mathbf{z}'), \xi')$  are two “perturbed” stable maps. We say they are equivalent if there is an isomorphism  $\theta : (\Sigma, \mathbf{z}) \rightarrow (\Sigma', \mathbf{z}')$  such that  $(\tilde{h}, \xi) = (\tilde{h}', \xi') \circ \theta$  and  $\theta^*(\bar{\partial}\tilde{h}') = \bar{\partial}\tilde{h}$ , where  $\theta^*$  is the pull-back map from  $(TX)_{\xi'}^* \otimes \Lambda^{0,1}(\Sigma')$  to  $(TX)_\xi^* \otimes \Lambda^{0,1}(\Sigma)$ . We can define a topology on the space of equivalence classes of “perturbed” stable maps in the same way as we did for the moduli space of stable maps.

Here we insert a lemma which will play the role of the unique continuity property for pseudo-holomorphic maps.

**Lemma 3.3.3:** *Let  $D$  be the unit disc in  $\mathbf{C}$ , and  $\tilde{f}_i$  for  $i = 1, 2$  be two  $C^\infty$  maps from  $D$  into a uniformizing system  $(V, G, \pi)$ . Then if  $\bar{\partial}\tilde{f}_1 = \bar{\partial}\tilde{f}_2$  and  $\tilde{f}_1 - \tilde{f}_2$  vanishes to an infinite order at 0, then  $\tilde{f}_1 \equiv \tilde{f}_2$ .*

**Proof:** Written down in local coordinates, the condition  $\bar{\partial} \tilde{f}_1 = \bar{\partial} \tilde{f}_2$  means

$$\frac{\partial u_1^\alpha}{\partial t} + J(u_1^\alpha) \frac{\partial u_1^\alpha}{\partial s} = \frac{\partial u_2^\alpha}{\partial t} + J(u_2^\alpha) \frac{\partial u_2^\alpha}{\partial s}$$

and

$$\frac{\partial u_1^\alpha}{\partial s} - J(u_1^\alpha) \frac{\partial u_1^\alpha}{\partial t} = \frac{\partial u_2^\alpha}{\partial s} - J(u_2^\alpha) \frac{\partial u_2^\alpha}{\partial t}$$

from which we can derive the condition needed in the Hartman-Wintner lemma

$$\|\Delta v\| \leq C(\|\partial_t v\| + \|\partial_s v\| + \|v\|)$$

where  $v = (u_1^\alpha - u_2^\alpha)$ . □

**Remark 3.3.4:** *There is a technical issue involved in the construction. For any two sections  $\tilde{s}_1$  and  $\tilde{s}_2$  near a stable map  $\tau$  that represent the same  $C^\infty$  map, we need to show that there is an element  $g \in G_\tau$  such that  $\tilde{s}_2 = g \cdot \tilde{s}_1$ . For this we need a certain unique continuity property, which fails for general  $C^\infty$  maps. This lemma tells us that we can overcome this problem by controlling the image under the  $\bar{\partial}$  operator, which lies in the obstruction bundle  $E$ . So we need to require that any obstruction bundle under consideration consists of sections such that if any local representatives of two sections are related by an isomorphism  $\delta$ , it must be induced from a global one, i.e., an element in  $G_\tau$ .* □

For each equivalence class of “perturbed” stable maps  $\kappa$ , we fix a representative  $(h_\kappa, (\Sigma, \mathbf{z})_\kappa, \xi_\kappa)$ . Suppose  $\kappa$  is close to  $\sigma_i$  so that there is a  $(y, \varsigma, \tilde{s})$  such that  $\kappa$  is represented by  $h_{(y, \varsigma, \tilde{s})} = \text{Exp} \circ \bar{f}_{i, (y, \varsigma)} \circ \tilde{s}$ . We add  $(\theta_{y, \varsigma} \circ \text{Par})^{-1}(\bar{\partial} h_{(y, \varsigma, \tilde{s})})$  to  $E_i$  and denote by  $\hat{E}_{i, \kappa}$  the finite dimensional  $\Gamma_i$ -invariant subspace generated by them. We repeat the local construction in the previous subsection with the choice of  $\hat{E}_{i, \kappa}$ , and we obtain finitely many,  $\Gamma_i$ -invariant representatives  $(y, \varsigma, \tilde{s})_j$  for  $\kappa$ . Then we can do the same thing to  $\kappa$  as we did to each stable map  $\tau$ . We use parallel transport along geodesics  $\text{Exp} \circ \bar{f}_{i, (y, \varsigma)} \circ t\tilde{s}_j(z)$  to transport the subspace  $E_{i, (y, \varsigma)_j}$  of  $L_\delta^p((TX)_{i, (y, \varsigma)_j}^* \otimes \Lambda^{0,1})$  into  $C^\infty((TX)_{(y, \varsigma, \tilde{s})_j}^* \otimes \Lambda^{0,1}(\Sigma_{y, \varsigma})_j)$ , where  $(TX)_{(y, \varsigma, \tilde{s})_j}^*$  is the pull-back orbifold bundle by  $\text{Exp} \circ \bar{f}_{i, (y, \varsigma)} \circ \tilde{s}_j$  over  $\Sigma_{(y, \varsigma)_j}$ . We denote the resulting subspace by  $E_{i, (y, \varsigma, \tilde{s})_j}$ . Then we have  $\gamma^*(E_{i, \gamma(y, \varsigma, \tilde{s})_j}) = E_{i, (y, \varsigma, \tilde{s})_j}$  for any  $\gamma \in \Gamma_i$ . Let  $\theta_{j, \kappa} : (\Sigma, \mathbf{z})_\kappa \rightarrow (\Sigma_{(y, \varsigma)_j}, \mathbf{z}_{(y, \varsigma)_j})$  be a biholomorphism sending marked points to marked points (not including new marked points on  $\Sigma_{(y, \varsigma)_j}$ ). Then the pull-back  $\theta_{j, \kappa}^*(E_{i, (y, \varsigma, \tilde{s})_j})$  in  $C^\infty((TX)_\kappa^* \otimes \Lambda^{0,1}(\Sigma_\kappa))$  does not depend on the choice of  $j$  and  $\theta_{j, \kappa}$ , where  $(TX)_\kappa^*$  is the pull-back orbifold bundle by  $(h_\kappa, (\Sigma, \mathbf{z})_\kappa, \xi_\kappa)$ . We denote  $\theta_\kappa^*(E_{i, (y, \varsigma, \tilde{s})_j})$  by  $E_{\kappa, i}$ . The above choice of representatives  $(y, \varsigma, \tilde{s})_j$  of  $\kappa$  are canonical in the sense that both of the following conditions are satisfied: they are  $\Gamma_i$ -invariant, and when  $\kappa$  converges to a stable map  $\tau \in O_i$ , the representatives of  $\kappa$  converge to the representatives of  $\tau$ . However, these representatives depend on the choice of  $(y, \varsigma, \tilde{s})$  at the beginning.

Now for each equivalence class of stable maps  $\tau$ , we consider all the equivalence classes of “perturbed” stable maps  $\kappa$  which are sufficiently close to  $\tau$ , and there is a representative  $(y, \varsigma, \tilde{s})$  centered at  $\tau$ . We denote this set by  $\mathcal{B}_\tau$  (we can think of it in the sense of germs). For each  $\kappa \in \mathcal{B}_\tau$ , we construct an obstruction space  $E_{\kappa, \tau}$  as a finite dimensional subspace in  $C^\infty((TX)_\kappa^* \otimes \Lambda^{0,1}(\Sigma_\kappa))$  as follows. Suppose  $\tau \in \cap_i \hat{O}_i$  for  $i = i_1, \dots, i_l$ . Then we define  $E_{\kappa, \tau}$  to be the span of  $E_{\kappa, i}$  for  $i = i_1, \dots, i_l$ . Note that  $E_{\kappa, i}$  are linearly independent when  $\kappa$  is sufficiently close to  $\tau$ . As an immediate consequence, which is crucial to the fulfillment of the compatibility requirement in the construction of Kuranishi structure, we have  $E_{\kappa, \tau_1} \subset E_{\kappa, \tau_2}$  when  $\tau_1$  is sufficiently close to  $\tau_2$ .

For each  $\tau \in \overline{\mathcal{M}}_{g, k}(X, J, A, \mathbf{x})$ , we define  $Z_\tau$  to be the solution set of  $Z_\tau = \{\kappa \in \mathcal{B}_\tau \mid \bar{\partial} h_\kappa \in E_{\kappa, \tau}\}$ . The main result of this subsection on the construction of Kuranishi structure is summarized in

**Proposition 3.3.5:** *Each  $Z_\tau$  is an orbifold and  $E_\tau/\Gamma_\tau$  is an orbifold bundle over  $Z_\tau$  with a continuous section  $s_\tau$  and a map  $\psi_\tau$  such that  $\psi_\tau$  is a homeomorphism from  $s_\tau^{-1}(0)$  onto a neighborhood of  $\tau$  in  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ , i.e.,  $(Z_\tau, E_\tau/\Gamma_\tau, s_\tau, \psi_\tau)$  is a Kuranishi neighborhood at  $\tau$ . Moreover, for any  $\sigma \in \overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  which is sufficiently close to  $\tau$ , we have an orbifold embedding  $(\phi_{\tau\sigma}, \hat{\phi}_{\tau\sigma})$  from  $(Z_\sigma, E_\sigma/\Gamma_\sigma)$  into  $(Z_\tau, E_\tau/\Gamma_\tau)$ , where  $\phi_{\tau\sigma}$  is given by the natural inclusion. There is also an orbifold bundle isomorphism  $\Phi_{\tau\sigma} : TZ_\tau/TZ_\sigma \rightarrow (E_\tau/\Gamma_\tau)/(E_\sigma/\Gamma_\sigma)$ , and the collection*

$$\{(Z_\tau, E_\tau/\Gamma_\tau, s_\tau, \psi_\tau, \phi_{\tau\sigma}, \hat{\phi}_{\tau\sigma}, \Phi_{\tau\sigma}) : \tau \in \overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})\}$$

*forms a stably complex Kuranishi structure on  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$  of dimension  $2d$  for*

$$d = c_1(TX) \cdot A + \left(\frac{1}{2} \dim_{\mathbf{R}} X - 3\right)(1 - g) + k - \iota(\mathbf{x}).$$

Here  $\iota(\mathbf{x}) = \sum_{i=1}^k \iota_{(g_i)}$  for  $\mathbf{x} = (X_{(g_1)}, \dots, X_{(g_k)})$  where  $\iota_{(g)}$  is the degree shifting number for  $X_{(g)}$ .

**Proof:** We first show that  $Z_\tau$  is an orbifold. Suppose that  $\tau$  is in  $O_i$ . For any  $(y, \varsigma, \tilde{s})$  centered at  $\tau$ , we construct a sequence  $(y, \varsigma, \tilde{s}_n)$  by cutting down  $\tilde{s}$  on the cylindrical ends such that  $\tilde{s}_n$  converges to  $\tilde{s}$ . Then the “perturbed” stable maps defined by  $(y, \varsigma, \tilde{s}_n)$  can be represented by some  $(y', \varsigma', \tilde{s}')_n$  centered at  $\sigma_i$  when  $\tilde{s}$  is small, and by repeating the local construction we obtain some canonical representatives which are  $\Gamma_i$ -invariant. When  $n \rightarrow \infty$ , these representatives converge to a set of  $\Gamma_i$ -invariant representatives for  $(y, \varsigma, \tilde{s})$ . Similar to what we did to a “perturbed” stable map, we obtain a finite dimensional subspace in  $C^\infty((TX)_{(y, \varsigma, \tilde{s})}^* \otimes \Lambda^{0,1})$ ; call it  $E_{(y, \varsigma, \tilde{s}), \tau}$ . If  $(y, \varsigma, \tilde{s})$  represents a “perturbed” stable map  $\kappa$ , then  $E_{(y, \varsigma, \tilde{s}), \tau}$  is just the pull-back of  $E_{\kappa, \tau}$ . This system of subspaces is also  $\Gamma_\tau$ -equivariant. We want to consider the solution set of  $\{(y, \varsigma, \tilde{s}) | \partial h_{(y, \varsigma, \tilde{s})} \in E_{(y, \varsigma, \tilde{s})}\}$ , where  $h_{(y, \varsigma, \tilde{s})}$  is the  $C^\infty$  map defined by  $(y, \varsigma, \tilde{s})$ . The technique of the local construction can be modified to show that this solution set (or a slice of it when  $(\Sigma, \mathbf{z})_\tau$  has an unstable component) is identified with a  $\Gamma_\tau$ -invariant finite dimensional open ball  $U_\tau$ , and  $U_\tau/\Gamma_\tau$  is homeomorphic to  $Z_\tau$ . Moreover,  $E_\tau/\Gamma_\tau$  is an orbifold bundle over  $Z_\tau$  with a continuous section  $s_\tau$  and a map  $\psi_\tau$  such that  $\psi_\tau$  is a homeomorphism from  $s_\tau^{-1}(0)$  onto a neighborhood of  $\tau$  in  $\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})$ . So we have shown that  $(Z_\tau, E_\tau/\Gamma_\tau, s_\tau, \psi_\tau)$  is a Kuranishi neighborhood at  $\tau$ . The dimension of the Kuranishi structure is easily seen to be independent of  $\tau$ .

As for the maps  $\phi_{\tau\sigma}, \hat{\phi}_{\tau\sigma}$  and  $\Phi_{\tau\sigma}$ , we observe that if  $\kappa \in \mathcal{B}_\sigma$ , this means that  $\kappa$  is represented by some  $(y, \varsigma, \tilde{s})$  with  $\tilde{s}$  sufficiently small. If  $\sigma$  is close enough to  $\tau$ ,  $\sigma$  is also represented by a  $C^\infty$  section, so that  $\kappa$  can be also represented by some  $(y', \varsigma', \tilde{s}')$  centered at  $\tau$ . This means that  $\mathcal{B}_\sigma$  is naturally included in  $\mathcal{B}_\tau$  when  $\sigma$  is sufficiently close to  $\tau$ . Then  $E_{\kappa, \sigma} \subset E_{\kappa, \tau}$  implies that  $Z_\sigma$  is naturally included in  $Z_\tau$ , which is defined to be  $\phi_{\tau\sigma}$ . In order to show that  $\phi_{\tau\sigma}$  is an orbifold embedding, we observe that the local group  $\Gamma_\sigma$  is isomorphic to the isotropy subgroup of any representative of  $\sigma$  in  $U_\tau$  (Lemma 3.3.1). This implies the embedding between the germs of the corresponding uniformizing systems, because they are totally determined by the local groups via geodesic charts. Similarly  $\hat{\phi}_{\tau\sigma}$  is obtained via natural inclusions  $E_{\kappa, \sigma} \subset E_{\kappa, \tau}$ . The map  $\Phi_{\tau\sigma}$  is obtained through the following consideration. Suppose that  $F$  is a nonlinear Fredholm map with linearization  $L$  at 0, and a Kuranishi model is constructed with a choice of obstruction space  $E_1$  and  $E_2$  respectively, on a small ball of 0 in  $V_1 = L^{-1}(E_1)$  and  $V_2 = L^{-1}(E_2)$  respectively. When  $E_1$  is a subspace of  $E_2$ ,  $V_1$  is a subspace of  $V_2$  too, and the complement of  $V_1$  in  $V_2$  is isomorphic to the complement of  $E_1$  in  $E_2$  via  $L$ . However, there is a slight complication here. When  $(\Sigma, \mathbf{z})_\tau$  has an unstable component, new marked points are added, and only a subspace of  $V$ , which is complementary to the direction of parametrization of the new marked points, is used. But when  $(\Sigma, \mathbf{z})_\sigma$  is a stable curve, there is no need for new marked points, and the whole space  $V$  is used. The remedy to this problem is this: We add new marked points to  $\sigma$  accordingly, and one can verify that the new Kuranishi

model is canonically isomorphic to the old one, because after adding new marked points, the space of  $V_{deform} \times V_{resolv}$  is “thickened” by the direction of parametrization of new marked points, which is isomorphic to the direction in  $V$  which is complementary to the subspace in  $V$  used in the new Kuranishi model. The compatibility requirement is fulfilled automatically because all of the maps are obtained through natural inclusions.

Finally, we show that the Kuranishi structure thus obtained is stably complex. The key point in this fact is that the symbol of the linearization of the Cauchy-Riemann equation is complex linear and the space  $V_{deform} \times V_{resolv}$  is naturally a complex manifold. Let’s first look at a toy model. Suppose we have an orbifold  $U$  and a smooth family of Fredholm operators  $F_s : X_s \rightarrow Y_s$  for  $s \in U$  which can be lifted to local charts of  $U$  equivariantly. If  $E \rightarrow U$  is a finite dimensional orbifold bundle over  $U$  and for each  $s \in U$  the fiber  $E_s \subset Y_s$  such that  $F_s : X_s \rightarrow Y_s/E_s$  is surjective, then  $V = \{V_s\}$  is an orbifold bundle over  $U$ , where  $V_s = F_s^{-1}(E_s)$ . If we deform  $F = \{F_s\}$  through a homotopy  $F^t = \{F_s^t\}$ , and let  $V^t = (F^t)^{-1}(E)$ , then  $V^t$  are isomorphic. In the case we don’t have an orbifold  $U$ , but a Kuranishi structure in general, we can do this on each Kuranishi neighborhood compatibly. In the present case, we take a complex linear vector space for each  $E_i$ , and use the complex linear part of the geodesic parallel transport to transport each  $E_i$  to nearby points, so we can assume each  $E_{\kappa,\tau}$  is complex linear. Over each Kuranishi neighborhood  $Z_\tau$ , the tangent bundle  $TZ_\tau$  is given by  $T(V_{deform} \times V_{resolv}/\Gamma_\tau) \oplus V_\tau$ , where  $V_\tau$  is an orbifold bundle obtained from the family of Fredholm operators  $D\bar{\partial}h_\kappa$  at  $([y, \varsigma], \kappa)$ , which can be deformed to a family of complex linear Fredholm operators since the symbols are complex linear. Hence  $TZ_\tau$  is stably isomorphic to a complex orbifold bundle. However, there is a slight complication here which does not effect the conclusion, namely, when  $(\Sigma, \mathbf{z})_\tau$  is a stable curve, we take  $V_\tau$  to be the orbifold bundle  $\{V_{([y, \varsigma], \kappa)} = (D\bar{\partial}h_\kappa)^{-1}(E_{\kappa,\tau})\}$ ; but when  $(\Sigma, \mathbf{z})_\tau$  has a unstable component, we take a subbundle of it which is stably complex. The complement is made up through the “thickening” in the space  $V_{deform} \times V_{resolv}$  by the parametrization of new marked points. See [FO] for more details.  $\square$

### 3.4 Orbifold Gromov-Witten invariants and axioms

Once we construct the Kuranishi structure with the necessary patching properties, we can use Fukaya-Ono’s abstract argument to construct a virtual fundamental cycle of the moduli space of orbifold stable maps in  $H_*(\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x}), \mathbf{Q})$ , denoted by  $[\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})]^{vir}$ . The degree of  $[\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})]^{vir}$  is given by the index formula  $2C_1(A) + 2n(3 - g) + 2k - 2\ell(\mathbf{x})$ .

For any component  $\mathbf{x} = (X_{(g_1)}, \dots, X_{(g_k)})$ , there are  $k$  evaluation maps (cf. (3.5))

$$(5.1) \quad e_i : \overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x}) \rightarrow X_{(g_i)}, \quad i = 1, \dots, k.$$

We also have a map

$$p : \overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x}) \rightarrow \overline{\mathcal{M}}_{g,k}$$

where  $p$  contracts the unstable components of the domain to obtain a stable Riemann surface in  $\overline{\mathcal{M}}_{g,k}$ . For any set of cohomology classes  $\alpha_i \in H^{*-2\ell(g_i)}(X_{(g_i)}; \mathbf{Q}) \subset H_{orb}^*(X; \mathbf{Q})$ ,  $i = 1, \dots, k$  and  $K \in H^*(\overline{\mathcal{M}}_{g,k}, \mathbf{Q})$ , the orbifold Gromov-Witten invariant is defined as the pairing

$$\Psi_{(g,k,A,\mathbf{x})}^{X,J}(K; \mathcal{O}_{l_1}(\alpha_1), \dots, \mathcal{O}_{l_k}(\alpha_k)) = p^*K \cup \prod_{i=1}^k c_1(L_i)^{l_i} e_i^* \alpha_i [\overline{\mathcal{M}}_{g,k}(X, J, A, \mathbf{x})]^{vir}.$$

where  $L_i$  is the line bundle generated by the cotangent space of the  $i$ -th marked point. When  $l_i < 0$ , we define it to be zero. The same argument as in the smooth case yields

**Proposition 3.4.1:**

1.  $\Psi_{(g,k,A,\mathbf{x})}^{X,J}(\mathcal{O}_{l_1}(K; \alpha_1), \dots, \mathcal{O}_{l_k}(\alpha_k)) = 0$  unless  $\deg K + \sum_i (\deg_{orb}(\alpha_i) + l_i) = 2C_1(A) + 2(n - 3)(1 - g) + 2k$ , where  $\deg_{orb}(\alpha_i)$  is the orbifold degree of  $\alpha_i$  obtained after degree shifting.
2.  $\Psi_{(g,k,A,\mathbf{x})}^{X,J}(K; \mathcal{O}_{l_1}(\alpha_1), \dots, \mathcal{O}_{l_k}(\alpha_k))$  is independent of the choice of  $J$  and hence is an invariant of symplectic structures.

We can drop  $J$  from the notation.

In the smooth case, Gromov-Witten invariants satisfy a set of axioms which serves as the guide line to construct general Gromov-Witten invariants. The deformation invariants axiom was first proposed by Ruan [Ru]. The others came from physics and were first proposed by Witten in [W]. The same argument as that of Fukaya-Ono also yields the same axioms for orbifold Gromov-Witten invariants except for a slightly restrictive divisor axiom. We shall list them here and refer for the proof to [FO]. Here, we list the axioms for the invariant without descendants only. The invariant with descendants ( $l_i > 0$ ) satisfies slightly modified axioms which is the same as in the smooth case. We leave it to the readers.

**Theorem 3.4.2:**  $\Psi_{(g,k,A,\mathbf{x})}^X(K; \alpha_1, \dots, \alpha_k)$  satisfies following axioms:

**Deformation Invariance Axiom**  $\Psi_{(g,k,A,\mathbf{x})}^X(K; \alpha_1, \dots, \alpha_k)$  is independent of the smooth deformation of symplectic structures.

There is a map  $\pi : \overline{\mathcal{M}}_{g,k+1} \rightarrow \overline{\mathcal{M}}_{g,k}$  for  $(g, k) \neq (0, 2), (1, 0)$ . This map yields two axioms.

**Point Axiom** For any  $\alpha_1, \dots, \alpha_k$  in  $H_{orb}^*(X, \mathbf{Q})$ , we have

$$\Psi_{(A,g,k+1)}^X(K; 1, \alpha_1, \dots, \alpha_k) = \Psi_{(A,g,k)}^X(\pi_* K; \alpha_1, \dots, \alpha_k)$$

**Divisor Axiom** Let  $\beta \in H^2(X, \mathbf{Q})$  (not  $H_{orb}^2(X, \mathbf{Q})$ ).

$$\Psi_{(A,g,k+1)}^X(\pi^* K; \beta, \alpha_1, \dots, \alpha_k) = \beta(A) \Psi_{(A,g,k)}^X(K; \alpha_1, \dots, \alpha_k).$$

There are two maps

$$\begin{aligned} \theta_1 : \overline{\mathcal{M}}_{g_1,k_1+1} \times \overline{\mathcal{M}}_{g_2,k_2+1} &\rightarrow \overline{\mathcal{M}}_{g_1+g_2,k_1+k_2}, \\ \theta_2 : \overline{\mathcal{M}}_{g,k+2} &\rightarrow \overline{\mathcal{M}}_{g+1,k} \end{aligned}$$

by joining the two extra marked points.  $\theta_1, \theta_2$  gives two axioms collectively called the splitting axiom. Let  $\Delta \subset \widehat{\Sigma X} \times \widehat{\Sigma X}$  be the graph of involution map  $I$ . Using Künneth formula, we can express its Poincaré dual as

$$\Delta^* = \sum_{a,b} \eta^{ab} \beta_a \otimes \beta_b,$$

where  $\beta_a$  is a basis of  $H_{orb}^*(X, \mathbf{Q})$ . Note that  $\eta^{ab} = \langle \beta_a, \beta_b \rangle_{orb}$ .

**Splitting Axiom I** Let  $K_i \in H^*(\overline{\mathcal{M}}_{g_i,k_i+1}, \mathbf{Q})$ . Then,

$$\begin{aligned} &\Psi_{(A,g_1+g_2,k_1+k_2)}^X((\theta_1)_*(K_1 \cup K_2)\{\alpha_i\}) \\ &= (-1)^{\deg(K_2) \sum_{i=1}^{k_1} \deg(\alpha_i)} \sum_{A=A_1+A_2} \sum_{a,b} \Psi_{(A_1,g_1,k_1+1)}^X(K_1; \{\alpha_i\}_{i \leq k}, \beta_a) \eta^{ab} \Psi_{(A_2,g_2,k_2+1)}^Y(K_2; \beta_b, \{\alpha_j\}_{j > k}) \end{aligned}$$

**Splitting Axiom II** Let  $K \in H^*(\overline{\mathcal{M}}_{g,k+2}, \mathbf{Q})$ .

$$\Psi_{(A,g+1,k)}^X((\theta_2)_*(K); \alpha_1, \dots, \alpha_k) = \sum_{a,b} \Psi_{(A,g-1,k+2)}^X(K; \alpha_1, \dots, \alpha_k, \beta_a, \beta_b) \eta^{ab}.$$

It is well-known that Gromov-Witten invariants satisfying the previous axioms yield an associative quantum multiplication over  $H_{orb}^*(X, \Lambda_\omega)$  where  $\Lambda_\omega$  is the Novikov ring. Define

$$\Psi^X(\alpha, \beta, \gamma) = \sum_A \Psi_{(A,0,3)}^X(1; \alpha, \beta, \gamma) q^A.$$

Then the quantum product  $\alpha \times_Q \beta$  is defined by the relation

$$\langle \alpha \times_Q \beta, \gamma \rangle_{orb} = \Psi^Y(\alpha, \beta, \gamma).$$

This is so called the "small" quantum product. One can also define the big quantum product which depends on a parameter  $w \in H_{orb}^*(X, \mathbf{C})$ . The definition is identical to the smooth case. We omit the details.

**Theorem 3.4.3:** *Quantum product (small or big) is associative.*

## 4 Appendix: An Introduction to Orbifolds

In this section we give some relevant background material on orbifolds. Some of it are scattered in the literature or folklore theorems; some are completely new (e.g. the notion of good map) or have no precise formulation available in the literature. It seems to us that there is no good reference on orbifold theory available, starting from the basics, so it is worth while to write up such a self-contained exposition in which almost everything has a detailed proof. We also provide many examples to assist in understanding the material. Our formulation is along the lines of [FO] in contrast to the more traditional ones in [S] or [K1].

### 4.1 Basic notions of orbifold

Primary examples of orbifolds are quotient spaces of smooth manifolds by a smooth finite group action. Here we consider that the quotient space is uniformized (or modeled) by the manifold with the finite group action. Hence a notion of smoothness for the quotient space is inherited from the manifold through those objects which are invariant under the group action. We require that any element of the group either acts trivially or has fixed-point set of codimension at least two. This is the case, for example, when the action is orientation-preserving. This requirement has a consequence that the non-fixed-point set is locally connected.

Let  $U$  be a connected topological space,  $V$  be a connected  $n$ -dimensional smooth manifold and  $G$  be a finite group acting on  $V$  smoothly. An  $n$ -dimensional uniformizing system of  $U$  is a triple  $(V, G, \pi)$ , where  $\pi : V \rightarrow U$  is a continuous map inducing a homeomorphism between  $V/G$  and  $U$ . Two uniformizing systems  $(V_i, G_i, \pi_i)$ ,  $i = 1, 2$ , are *isomorphic* if there is a diffeomorphism  $\phi : V_1 \rightarrow V_2$  and an isomorphism  $\lambda : G_1 \rightarrow G_2$  such that  $\phi$  is  $\lambda$ -equivariant, and  $\pi_2 \circ \phi = \pi_1$ . It is easily seen that if  $(\phi, \lambda)$  is an automorphism of  $(V, G, \pi)$ , then there is a  $g \in G$  such that  $\phi(x) = g \cdot x$  and  $\lambda(a) = g \cdot a \cdot g^{-1}$  for any  $x \in V$  and  $a \in G$ .  $g$  is unique iff the action of  $G$  on  $V$  is effective. We use  $\ker G$  to denote the subgroup of  $G$  acting trivially.

Let  $i : U' \hookrightarrow U$  be a connected open subset of  $U$ , and  $(V', G', \pi')$  be a uniformizing system of  $U'$ . We say that  $(V', G', \pi')$  is induced from  $(V, G, \pi)$  if there is a monomorphism  $\lambda : G' \rightarrow G$  and a  $\lambda$ -equivariant open embedding  $\phi : V' \rightarrow V$  such that  $\lambda$  induces an isomorphism from  $\ker G'$  to  $\ker G$  and  $i \circ \pi' = \pi \circ \phi$ . We follow Satake [S] and call  $(\phi, \lambda) : (V', G', \pi') \rightarrow (V, G, \pi)$  an *injection*. Two injections  $(\phi_i, \lambda_i) : (V'_i, G'_i, \pi'_i) \rightarrow (V, G, \pi)$ ,  $i = 1, 2$ , are *isomorphic* if there is an isomorphism  $(\psi, \tau)$  between  $(V'_1, G'_1, \pi'_1)$  and  $(V'_2, G'_2, \pi'_2)$ , and an automorphism  $(\bar{\psi}, \bar{\tau})$  of  $(V, G, \pi)$  such that  $(\bar{\psi}, \bar{\tau}) \circ (\phi_1, \lambda_1) = (\phi_2, \lambda_2) \circ (\psi, \tau)$ .

**Lemma 4.1.1:** *Let  $(V, G, \pi)$  be a uniformizing system of  $U$ . For any connected open subset  $U'$  of  $U$ ,  $(V, G, \pi)$  induces a unique isomorphism class of uniformizing systems of  $U'$ .*

**Proof:**

*Existence:* Consider the preimage  $\pi^{-1}(U')$  in  $V$ .  $G$  acts as permutations on the set of connected components of  $\pi^{-1}(U')$ . Let  $V'$  be one of the connected components of  $\pi^{-1}(U')$ ,  $G'$  be the subgroup of  $G$  which fixes the component  $V'$  and  $\pi' = \pi|_{V'}$ . Then  $(V', G', \pi')$  is an induced uniformizing system of  $U'$ .

*Uniqueness:* First of all, different choices of the connected components of  $\pi^{-1}(U')$  induce isomorphic uniformizing systems. Secondly, let  $(V'_1, G'_1, \pi'_1)$  be any induced uniformizing system of  $U'$  and  $(\psi, \tau)$  be the injection into  $(V, G, \pi)$ . We will show that  $(\psi, \tau)$  induces an isomorphism between  $(V'_1, G'_1, \pi'_1)$  and the induced uniformizing system given by a connected component of  $\pi^{-1}(U')$ . Suppose  $\psi(V'_1)$  lies in the connected component  $V'$ . We can show that  $\psi(V'_1)$  is closed in  $V'$ . Let  $\psi(x_n) \rightarrow y_0$  in  $V'$ ,  $x_n \in V'_1$ , then there exists a  $z_0 \in V'_1$  such that  $\pi'_1(z_0) = \pi(y_0)$ , and  $z_n \in V'_1$  such that  $z_n \rightarrow z_0$ ,  $\pi'_1(z_n) = \pi(\psi(x_n)) = \pi'_1(x_n)$ . So there exist  $a_n \in G'_1$  such that  $a_n(z_n) = x_n$ . Since  $G'_1$  is finite, it follows that for large  $n$ ,  $a_n = a$  is a constant. So  $x_n \rightarrow a(z_0)$  in  $V'_1$  and  $y_0 = \psi(a(z_0))$ , i.e.,  $\psi(V'_1)$  is closed in  $V'$ .  $\psi(V'_1)$  is also open in  $V'$ . So  $\psi$  induces a diffeomorphism between  $V'_1$  and  $V'$ . From this we can easily see that  $(V'_1, G'_1, \pi'_1)$  and  $(V', G', \pi')$  are isomorphic.  $\square$

Let  $U$  be a connected and locally connected topological space. For any point  $p \in U$ , we can define the *germ* of uniformizing systems at  $p$  in the following sense. Let  $(V_1, G_1, \pi_1)$  and  $(V_2, G_2, \pi_2)$  be uniformizing systems of neighborhoods  $U_1$  and  $U_2$  of  $p$ . We say that  $(V_1, G_1, \pi_1)$  and  $(V_2, G_2, \pi_2)$  are *equivalent* at  $p$  if they induce isomorphic uniformizing systems for a neighborhood  $U_3$  of  $p$ .

**Definition 4.1.2:** *Let  $X$  be a Hausdorff, second countable topological space. An  $n$ -dimensional orbifold structure on  $X$  is given by the following data: for any point  $p \in X$ , there is a neighborhood  $U_p$  and an  $n$ -dimensional uniformizing system  $(V_p, G_p, \pi_p)$  of  $U_p$  such that for any point  $q \in U_p$ ,  $(V_p, G_p, \pi_p)$  and  $(V_q, G_q, \pi_q)$  are equivalent at  $q$  (i.e., define the same germ at  $q$ ). The germ of orbifold structures on  $X$  is defined in the following sense: two orbifold structures  $\{(V_p, G_p, \pi_p) : p \in X\}$  and  $\{(V'_p, G'_p, \pi'_p) : p \in X\}$  are equivalent if for any  $p \in X$ ,  $(V_p, G_p, \pi_p)$  and  $(V'_p, G'_p, \pi'_p)$  are equivalent at  $p$ . With a given germ of orbifold structure on it,  $X$  is called a orbifold. We call each  $U_p$  a uniformized neighborhood of  $p$ , and  $(V_p, G_p, \pi_p)$  a chart at  $p$ . An open subset  $U$  of  $X$  is called an uniformized open set if it is uniformized by  $(V, G, \pi)$  such that for each  $p \in U$ ,  $(V, G, \pi)$  defines the same germ as  $(V_p, G_p, \pi_p)$  at  $p$ . Later on we will show and therefore always assume that for each  $p \in X$ ,  $(V_p, G_p, \pi_p)$  can be taken such that  $V_p$  is a ball in  $\mathbf{R}^n$ , and  $\pi_p$  sends the origin of  $\mathbf{R}^n$  to  $p$ . A point  $p \in X$  is called regular or smooth if  $G_p$  is trivial; otherwise, it is called singular. The set of smooth points is denoted by  $X_{reg}$ , and the set of singular points is denoted by  $\Sigma X$ . If  $G_p$  acts effectively, we call  $X$  a reduced orbifold.*

It is clear that every orbifold  $X$  induces a reduced orbifold  $X_{red}$  by redefining the local group. We call  $X_{red}$  the reduced associate of  $X$ .

**Remark 4.1.3:** *There is a notion of orbifold with boundary, in which we allow the uniformizing systems to be smooth manifolds with boundary, with a finite group action preserving the boundary. If  $X$  is an orbifold with boundary, then it is easily seen that the boundary  $\partial X$  inherits an orbifold structure from  $X$  and becomes an orbifold.*

**Example 4.1.4:** *Let's consider the 2-dimensional sphere  $S^2$ . Let  $D_s, D_n$  be open disc neighborhoods of the south pole and the north pole such that  $S^2 = D_s \cup D_n$ . Let  $D_s$  be uniformized by  $(\tilde{D}_s, \mathbf{Z}_2, \pi_s)$ , and  $D_n$  be uniformized by  $(\tilde{D}_n, \mathbf{Z}_3, \pi_n)$  where  $\mathbf{Z}_2, \mathbf{Z}_3$  act on  $\tilde{D}_s$  and  $\tilde{D}_n$  by rotations. For any point in  $S^2$  other than the south pole and the north pole, we take a chart at it induced by either*

$(\tilde{D}_s, \mathbf{Z}_2, \pi_s)$  or  $(\tilde{D}_n, \mathbf{Z}_3, \pi_n)$ . It is easily seen that this defines a 2-dimensional orbifold structure on  $S^2$ . Note that as an open subset of both  $D_s$  and  $D_n$ ,  $D_s \cap D_n$  has non-isomorphic induced uniformizing systems from  $(\tilde{D}_s, \mathbf{Z}_2, \pi_s)$  and  $(\tilde{D}_n, \mathbf{Z}_3, \pi_n)$ , although they define the same germ at each point in  $D_s \cap D_n$ . This also shows that although both  $D_s$  and  $D_n$  are uniformized, their union  $S^2$  cannot be uniformized, therefore is not a global quotient.

The notion of orbifold was first introduced by Satake in [S], where a different name,  $V$ -manifold, was used. In [S], an orbifold structure on a topological space  $X$  is given by an open cover  $\mathcal{U}$  of  $X$  satisfying the following conditions:

- (4.1.1a) Each element  $U$  in  $\mathcal{U}$  is uniformized, say by  $(V, G, \pi)$ .
- (4.1.1b) If  $U' \subset U$ , then there is a collection of injections  $(V', G', \pi') \rightarrow (V, G, \pi)$ .
- (4.1.1c) For any point  $p \in U_1 \cap U_2$ ,  $U_1, U_2 \in \mathcal{U}$ , there is a  $U_3 \in \mathcal{U}$  such that  $p \in U_3 \subset U_1 \cap U_2$ .

One can show that our definition is equivalent to Satake's. On the one hand, it is easy to see that an open cover of  $X$  satisfying (4.1.1a – c) gives rise to an orbifold structure on  $X$  in the sense of Definition 4.1.2. (We will call such a cover of an orbifold  $X$  a *compatible cover* if it gives rise to the same germ of orbifold structures on  $X$ .) On the other hand, given an orbifold  $X$  as in Definition 4.1.2, we can construct a compatible cover  $\mathcal{U}$  of  $X$  as follows: Take a locally finite refinement of  $\{U_p\}$ , denoted by  $\{U_i\}$ . Each  $q$  in  $X$  is contained in finitely many, say  $\alpha(q)$ , of  $U_i$ 's, so there is a connected open neighborhood  $W_q$  such that  $W_q$  is contained in these  $\alpha(q)$   $U_i$ 's and inherits a unique isomorphism class of uniformizing systems from them. Each connected open subset of  $W_q$  containing  $q$  inherits a uniformizing system from  $W_q$ . We call  $\alpha(q)$  the order of  $W_q$  or any of its open subsets containing  $q$ . Then we take  $\mathcal{U}$  to be the collection of all of the  $W_q$ 's and its connected open subsets containing  $q$ . We need to show that for any  $U_1$  and  $U_2$  in  $\mathcal{U}$  such that  $U_1 \subset U_2$ , there is an injection between the uniformizing systems  $(V_1, G_1, \pi_1) \rightarrow (V_2, G_2, \pi_2)$ . But this is easily seen from the fact that the order of  $U_1$  is always greater than or equal to the order of  $U_2$ .

Next we consider a class of continuous maps between two orbifolds which carry an additional structure of differentiability with respect to the orbifold structures. Let  $U$  be uniformized by  $(V, G, \pi)$  and  $U'$  by  $(V', G', \pi')$ , and  $f : U \rightarrow U'$  be a continuous map. A  $C^l$  *lifting*,  $0 \leq l \leq \infty$ , of  $f$  is a  $C^l$  map  $\tilde{f} : V \rightarrow V'$  such that  $\pi' \circ \tilde{f} = f \circ \pi$ , and for any  $g \in G$ , there is  $g' \in G'$  satisfying  $g' \cdot \tilde{f}(x) = \tilde{f}(g \cdot x)$  for any  $x \in V$ . Two liftings  $\tilde{f}_i : (V_i, G_i, \pi_i) \rightarrow (V'_i, G'_i, \pi'_i)$ ,  $i = 1, 2$ , are *isomorphic* if there exist isomorphisms  $(\phi, \tau) : (V_1, G_1, \pi_1) \rightarrow (V_2, G_2, \pi_2)$  and  $(\phi', \tau') : (V'_1, G'_1, \pi'_1) \rightarrow (V'_2, G'_2, \pi'_2)$  such that  $\phi' \circ \tilde{f}_1 = \tilde{f}_2 \circ \phi$ . Let  $p \in U$ , for any uniformized neighborhood  $U_p$  of  $p$  and uniformized neighborhood  $U_{f(p)}$  of  $f(p)$  such that  $f(U_p) \subset U_{f(p)}$ ; a lifting  $\tilde{f}$  of  $f$  will induce a lifting  $\tilde{f}_p$  for  $f|_{U_p} : U_p \rightarrow U_{f(p)}$  as follows: For any injection  $(\phi, \tau) : (V_p, G_p, \pi_p) \rightarrow (V, G, \pi)$ , consider the map  $\tilde{f} \circ \phi : V_p \rightarrow V'$ , and observe that  $\pi' \circ \tilde{f} \circ \phi(V_p) \subset U_{f(p)}$  implies  $\tilde{f} \circ \phi(V_p) \subset (\pi')^{-1}(U_{f(p)})$ . Therefore there is an injection  $(\phi', \tau') : (V_{f(p)}, G_{f(p)}, \pi_{f(p)}) \rightarrow (V', G', \pi')$  such that  $\tilde{f} \circ \phi(V_p) \subset \phi'(V_{f(p)})$ . We define  $\tilde{f}_p = (\phi')^{-1} \circ \tilde{f} \circ \phi$ . In this way we obtain a lifting  $\tilde{f}_p : (V_p, G_p, \pi_p) \rightarrow (V_{f(p)}, G_{f(p)}, \pi_{f(p)})$  for  $f|_{U_p} : U_p \rightarrow U_{f(p)}$ . We can verify that different choices give isomorphic liftings. We define the *germ* of liftings as follows: two liftings are *equivalent at  $p$*  if they induce isomorphic liftings on a smaller neighborhood of  $p$ .

Now consider orbifolds  $X$  and  $X'$  and a continuous map  $f : X \rightarrow X'$ . A *lifting* of  $f$  consists of the following data: for any point  $p \in X$ , there exist charts  $(V_p, G_p, \pi_p)$  at  $p$  and  $(V_{f(p)}, G_{f(p)}, \pi_{f(p)})$  at  $f(p)$  and a lifting  $\tilde{f}_p$  of  $f|_{\pi_p(V_p)} : \pi_p(V_p) \rightarrow \pi_{f(p)}(V_{f(p)})$  such that for any  $q \in \pi_p(V_p)$ ,  $\tilde{f}_p$  and  $\tilde{f}_q$  induce the same germ of liftings of  $f$  at  $q$ . We can define the *germ* of liftings in the sense that two



liftings of  $f$   $\{\tilde{f}_{p,i} : (V_{p,i}, G_{p,i}, \pi_{p,i}) \rightarrow (V_{f(p),i}, G_{f(p),i}, \pi_{f(p),i}) : p \in X\}$ ,  $i = 1, 2$ , are *equivalent* if for each  $p \in X$ ,  $\tilde{f}_{p,i}$ ,  $i = 1, 2$ , induce the same germ of liftings of  $f$  at  $p$ .

**Definition 4.1.5:** A  $C^l$  map ( $0 \leq l \leq \infty$ ) between orbifolds  $X$  and  $X'$  is a germ of  $C^l$  liftings of a continuous map between  $X$  and  $X'$ . We denote by  $\tilde{f}$  a  $C^l$  map which is a germ of liftings of a continuous map  $f$ .

A sequence of  $C^l$  maps  $\tilde{f}_n$  is said to converge to a  $C^l$  map  $\tilde{f}_0$  in the  $C^l$  topology if there exists a sequence of liftings  $\tilde{f}_{p,n} : (V_p, G_p, \pi_p) \rightarrow (V_{f_n(p)}, G_{f_n(p)}, \pi_{f_n(p)})$  defining the germs  $\tilde{f}_n$  such that for each  $p \in X$ , there exists a chart  $(V_{f_0(p)}, G_{f_0(p)}, \pi_{f_0(p)})$  and an integer  $n(p) > 0$  with the following property: for each  $n \geq n(p)$ , there is an injection  $(\psi_{p,n}, \tau_{p,n}) : (V_{f_n(p)}, G_{f_n(p)}, \pi_{f_n(p)}) \rightarrow (V_{f_0(p)}, G_{f_0(p)}, \pi_{f_0(p)})$  such that  $\psi_{p,n} \circ \tilde{f}_{p,n}$  converges in  $C^l$  to  $\tilde{f}_{p,0}$  which defines the germ  $\tilde{f}_0$ .

**Example 4.1.6a:** The real line  $\mathbf{R}$  as a smooth manifold is trivially an orbifold. A  $C^l$  map from an orbifold  $X$  to  $\mathbf{R}$  is called a  $C^l$  function on  $X$ . The set of all  $C^l$  functions on  $X$  is denoted by  $C^l(X)$ . A  $C^l$  function is essentially a continuous function which locally can be lifted to an invariant  $C^l$  function on a local chart. On the other hand, a  $C^l$  map from  $\mathbf{R}$  (or an interval  $I$ ) into  $X$  is called a  $C^l$  path in  $X$ . In subsection 4.1.2, we will discuss geodesics on a Riemannian orbifold and define the exponential map.

**Example 4.1.6b:** Let  $X = \mathbf{R} \times \mathbf{C}$ , and be given an orbifold structure by  $(\mathbf{R} \times \mathbf{C}, \mathbf{Z}_4, \pi)$  where  $\mathbf{Z}_4$  acts only on the factor  $\mathbf{C}$  by multiplication of  $\sqrt{-1}$ . Define  $C^1$  maps  $\tilde{f}_1 : \mathbf{R} \rightarrow (\mathbf{R} \times \mathbf{C}, \mathbf{Z}_4, \pi)$  by  $t \rightarrow (t, t^2)$  and  $\tilde{f}_2 : \mathbf{R} \rightarrow (\mathbf{R} \times \mathbf{C}, \mathbf{Z}_4, \pi)$  by  $t \rightarrow (t, t^2)$  for  $t \leq 0$  and  $(t, \sqrt{-1}t^2)$  for  $t \geq 0$ . Then  $\tilde{f}_1, \tilde{f}_2$  induce the same continuous map  $f : \mathbf{R} \rightarrow X$ , but they are not isomorphic as  $C^1$  maps.

Next we describe the notion of orbifold bundle which corresponds to the notion of smooth vector bundle. We begin with local uniformizing systems for orbibundles. Given a uniformized topological space  $U$  and a topological space  $E$  with a surjective continuous map  $pr : E \rightarrow U$ , a *uniformizing system of rank  $k$  orbifold bundle* for  $E$  over  $U$  consists of the following data:

- A uniformizing system  $(V, G, \pi)$  of  $U$ .
- A uniformizing system  $(V \times \mathbf{R}^k, G, \tilde{\pi})$  for  $E$ . The action of  $G$  on  $V \times \mathbf{R}^k$  is an extension of the action of  $G$  on  $V$  given by  $g(x, v) = (gx, \rho(x, g)v)$ , where  $\rho : V \times G \rightarrow \text{Aut}(\mathbf{R}^k)$  is a smooth map satisfying:

$$\rho(gx, h) \circ \rho(x, g) = \rho(x, h \circ g), \quad g, h \in G, x \in V.$$

- The natural projection map  $\tilde{pr} : V \times \mathbf{R}^k \rightarrow V$  satisfies  $\pi \circ \tilde{pr} = pr \circ \tilde{\pi}$ .

We can similarly define *isomorphisms* between uniformizing systems of orbifold bundle for  $E$  over  $U$ . The only additional requirement is that the diffeomorphisms between  $V \times \mathbf{R}^k$  are linear on each fiber of  $\tilde{pr} : V \times \mathbf{R}^k \rightarrow V$ . Moreover, for each connected open subset  $U'$  of  $U$ , we can similarly prove that there is a unique isomorphism class of induced uniformizing systems of orbifold bundle for  $E' = pr^{-1}(U')$  over  $U'$ . The *germ* of uniformizing systems of orbifold bundle at a point  $p \in U$  can also be similarly defined.

**Definition 4.1.7:** Let  $X$  be an orbifold and  $E$  be a topological space with a surjective continuous map  $pr : E \rightarrow X$ . A rank  $k$  orbifold bundle structure on  $E$  over  $X$  consists of the following data: For each point  $p \in X$ , there is a uniformized neighborhood  $U_p$  and a uniformizing system of rank  $k$  orbifold bundle for  $pr^{-1}(U_p)$  over  $U_p$  such that for any  $q \in U_p$ , the uniformizing systems of orbifold bundle over  $U_p$  and  $U_q$  define the same germ at  $q$ . The germ of rank  $k$  orbifold bundle

structures on  $E$  over  $X$  can be similarly defined. The topological space  $E$  with a given germ of orbifold bundle structures becomes an orbifold and is called an orbifold bundle over  $X$ . Each chart  $(V_p \times \mathbf{R}^k, G_p, \tilde{\pi}_p)$  is called a local trivialization of  $E$ . At each point  $p \in X$ , the fiber  $E_p = \text{pr}^{-1}(p)$  is isomorphic to  $\mathbf{R}^k/G_p$ . It contains a linear subspace  $E^p$  of fixed points of  $G_p$ . Two orbibundles  $\text{pr}_1 : E_1 \rightarrow X$  and  $\text{pr}_2 : E_2 \rightarrow X$  are isomorphic if there is a  $C^\infty$  map  $\psi : E_1 \rightarrow E_2$  given by  $\tilde{\psi}_p : (V_{1,p} \times \mathbf{R}^k, G_{1,p}, \tilde{\pi}_{1,p}) \rightarrow (V_{2,p} \times \mathbf{R}^k, G_{2,p}, \tilde{\pi}_{2,p})$  which induces an isomorphism between  $(V_{1,p}, G_{1,p}, \pi_{1,p})$  and  $(V_{2,p}, G_{2,p}, \pi_{2,p})$ , and is a linear isomorphism between the fibers of  $\tilde{\text{pr}}_{1,p}$  and  $\tilde{\text{pr}}_{2,p}$ . By replacing  $\mathbf{R}^k$  with  $\mathbf{C}^k$ , we have the definition of complex orbifold bundle.

**Remark 4.1.8a:** There is a notion of orbifold bundle over an orbifold with boundary. One can easily verify that if  $\text{pr} : E \rightarrow X$  is an orbifold bundle over an orbifold with boundary  $X$ , then the restriction to the boundary  $\partial X$ ,  $E_{\partial X} = \text{pr}^{-1}(\partial X)$ , is an orbifold bundle over  $\partial X$ .

**Remark 4.1.8b:** One can define an orbifold bundle with fiber a general space in the same vein.

A  $C^l$  map  $\tilde{s}$  from  $X$  to an orbifold bundle  $\text{pr} : E \rightarrow X$  is called a  $C^l$  section if locally  $\tilde{s}$  is given by  $\tilde{s}_p : V_p \rightarrow V_p \times \mathbf{R}^k$  where  $\tilde{s}_p$  is  $G_p$ -equivariant and  $\tilde{\text{pr}} \circ \tilde{s}_p = \text{Id}$  on  $V_p$ . We observe that

- For each point  $p$ ,  $s(p)$  lies in  $E^p$ , the linear subspace of fixed points of  $G_p$ .
- The space of all  $C^l$  sections of  $E$ , denoted by  $C^l(E)$ , has a structure of vector space over  $\mathbf{R}$  (or  $\mathbf{C}$ ) as well as a  $C^l(X)$ -module structure.
- The  $C^l$  sections  $\tilde{s}$  are in 1 : 1 correspondence with the underlying continuous maps  $s$ .

Orbibundles are more conveniently described by transition maps (see [S]). More precisely, an orbifold bundle over an orbifold  $X$  can be constructed from the following data: A compatible cover  $\mathcal{U}$  of  $X$  such that for any injection  $i : (V', G', \pi') \rightarrow (V, G, \pi)$ , there is a smooth map  $g_i : V' \rightarrow \text{Aut}(\mathbf{R}^k)$  giving an open embedding  $V' \times \mathbf{R}^k \rightarrow V \times \mathbf{R}^k$  by  $(x, v) \rightarrow (i(x), g_i(x)v)$ , and for any composition of injections  $j \circ i$ , we have

$$(4.1.2) \quad g_{j \circ i}(x) = g_j(i(x)) \circ g_i(x), \forall x \in V.$$

Two collections of maps  $g^{(1)}$  and  $g^{(2)}$  define isomorphic orbibundles if there are maps  $\delta_V : V \rightarrow \text{Aut}(\mathbf{R}^k)$  such that for any injection  $i : (V', G', \pi') \rightarrow (V, G, \pi)$ , we have

$$(4.1.3) \quad g_i^{(2)}(x) = \delta_V(i(x)) \circ g_i^{(1)}(x) \circ (\delta_{V'}(x))^{-1}, \forall x \in V'.$$

Since (4.1.2) behaves naturally under constructions of vector spaces such as tensor product, exterior product, etc. we can define these constructions for orbibundles.

**Example 4.1.10:** For an orbifold  $X$ , the tangent bundle  $TX$  can be constructed because the differential of any injection satisfies (4.1.2). Likewise, we define the cotangent bundle  $T^*X$ , the bundles of exterior power or tensor product. The  $C^\infty$  sections of these bundles give us vector fields, differential forms or tensor fields on  $X$ . There exists a de Rham cohomology theory for orbifolds, which is isomorphic to the singular cohomology theory of the underlying topological space. Observe also that if  $\omega$  is a differential form on  $X'$  and  $\tilde{f} : X \rightarrow X'$  is a  $C^\infty$  map, then there is a pull-back form  $\tilde{f}^*\omega$  on  $X$ .

Let  $U$  be an open subset of an orbifold  $X$  with an orbifold structure  $\{(V_p, G_p, \pi_p) : p \in X\}$ ; then  $\{(V'_p, G'_p, \pi'_p) : p \in U\}$  is an orbifold structure on  $U$ , where  $(V'_p, G'_p, \pi'_p)$  is a uniformizing system of  $\pi_p(V_p) \cap U$  induced from  $(V_p, G_p, \pi_p)$ . Likewise, let  $\text{pr} : E \rightarrow X$  be an orbifold bundle and  $U$

an open subset of  $X$ ; then  $pr : E_U = pr^{-1}(U) \rightarrow U$  inherits a unique germ of orbifold bundle structures from  $E$ , called the *restriction of  $E$  over  $U$* . When  $U$  is a uniformized open set in  $X$ , say uniformized by  $(V, G, \pi)$ , then there is a smooth vector bundle  $E_V$  over  $V$  with a smooth action of  $G$  such that  $(E_V, G, \tilde{\pi})$  uniformizes  $E_U$ . This is seen as follows: We first take a compatible cover  $\mathcal{U}$  of  $U$ , fine enough so that the preimage under  $\pi$  is a compatible cover of  $V$ . Let  $E_U$  be given by a set of transition maps with respect to  $\mathcal{U}$  satisfying (4.1.2); then the pull-backs under  $\pi$  form a set of transition maps with respect to  $\pi^{-1}(\mathcal{U})$  with an action of  $G$  by permutations, also satisfying (4.1.2), so that it defines a smooth vector bundle over  $V$  with a compatible smooth action of  $G$ . Any  $C^l$  section of  $E$  on  $X$  restricts to a  $C^l$  section of  $E_U$  on  $U$ , and when  $U$  is a uniformized open set by  $(V, G, \pi)$ , it lifts to a  $G$ -equivariant  $C^l$  section of  $E_V$  on  $V$ .

We end this subsection with a result which is analogous to the homotopy invariance of vector bundles. Let  $I = [0, 1]$ . Then if  $X$  is an orbifold,  $X \times I$  is an orbifold with boundary, with  $\partial(X \times I) = X \times \{0\} \cup X \times \{1\}$ .

**Proposition 4.1.11:** *Let  $pr : E \rightarrow X \times I$  be an orbifold bundle over  $X \times I$ . Then there is a  $C^\infty$  map  $\tilde{\Psi} : E \rightarrow E$  covering the  $C^\infty$  map  $\tilde{\psi} : X \times I \rightarrow X \times I$ , given by  $\tilde{\psi}(x, t) = (x, 1)$ , such that each local lifting of  $\tilde{\Psi}$  is an isomorphism on each fiber of  $\tilde{pr}$ . In particular, the two orbifold bundles  $E_{X \times \{0\}}$  and  $E_{X \times \{1\}}$  are isomorphic.*

**Proof:** First of all, by the compactness of  $I$  and the local triviality of  $E$ , for any  $p \in X$ , there exists a finite open covering  $\{I_i\}$  of  $I$  and a set of neighborhoods  $\{U_{p,i}\}$  of  $p$  such that  $E_{U_{p,i} \times I_i}$  is trivial. Take a neighborhood  $U_p \subset \cap_i U_{p,i}$ , then  $E_{U_p \times I_i}$  is trivial for all  $I_i$ 's. Next we show that if  $I_i \cap I_j \neq \emptyset$ , we can construct a trivialization of  $E_{U_p \times (I_i \cup I_j)}$ , which is a uniformizing system of  $E_{U_p \times (I_i \cup I_j)}$ , and a trivialization of  $E_{U_p \times I}$  successively. Without loss of generality, we assume that  $I_i = [0, b]$  and  $I_j = (a, 1]$ , for some  $a < b$ .

Let  $(V_p \times [0, b) \times \mathbf{R}^k, G_p, \tilde{\pi}_b)$  be a trivialization of  $E_{U_p \times [0, b)}$  and  $(V_p \times (a, 1] \times \mathbf{R}^k, G_p, \tilde{\pi}_a)$  a trivialization of  $E_{U_p \times (a, 1]}$ . We let  $(V_p \times (a, b) \times \mathbf{R}^k, G_p, \tilde{\pi}_{a,b})$  be the trivialization of  $E_{U_p \times (a, b)}$  induced from  $(V_p \times [0, b) \times \mathbf{R}^k, G_p, \tilde{\pi}_b)$ , and  $\psi : (V_p \times (a, b) \times \mathbf{R}^k, G_p, \tilde{\pi}_{a,b}) \rightarrow (V_p \times (a, 1] \times \mathbf{R}^k, G_p, \tilde{\pi}_a)$  be the injection covering the injection  $(V_p \times (a, b), G_p, \pi_{a,b}) \rightarrow (V_p \times (a, 1], G_p, \pi_a)$ , which is identical on the  $V_p$  factor. We remark that one can assume that the action of  $G_p$  on  $\mathbf{R}^k$  is independent of  $t \in I$ . Now we can define a trivialization  $(V_p \times [0, 1] \times \mathbf{R}^k, G_p, \tilde{\pi})$  of  $E_{U_p \times [0, 1]}$  as follows: pick a  $c$  satisfying  $a < c < b$  and a  $C^\infty$  diffeomorphism  $\beta : (a, 1] \rightarrow (a, b)$  which is the identity on  $(a, c)$ ; then we define

$$\tilde{\pi}(x, t, v) = \begin{cases} \tilde{\pi}_b(x, t, v) & \text{if } t \in [0, b) \\ \tilde{\pi}_a \circ (id \times \beta^{-1} \times id) \circ \psi^{-1} \circ (id \times \beta \times id)(x, t, v) & \text{if } t \in (a, 1]. \end{cases}$$

Now we pick a smaller neighborhood  $U'_p$  of  $p$  such that  $\overline{U'_p} \subset U_p$ , and a  $G_p$ -equivariant  $C^\infty$  function  $\theta(x, t)$  from  $V_p \times [0, 1]$  to  $[0, 1]$  such that  $\theta(x, t) = t$  for  $x$  outside a neighborhood of  $V'_p \subset V_p$  and  $\theta(x, t) = 1$  for  $x \in V'_p$ . We define  $\tilde{f}_p : E_{U_p \times I} \rightarrow E_{U_p \times I}$  to be the  $C^\infty$  map given by the  $G_p$ -equivariant map  $(x, t, v) \rightarrow (x, \theta(x, t), v)$ , which is the identity outside a neighborhood of  $V'_p \times [0, 1] \times \mathbf{R}^k$ . We can extend it to a  $C^\infty$  map  $E \rightarrow E$ , still denoted by  $\tilde{f}_p$ . Now we cover  $X$  by a locally finite covering of  $U'_p \subset U_{p_i}$ , and define  $\tilde{\Psi} : E \rightarrow E$  to be the product of the  $\tilde{f}_{p_i}$ 's.  $\square$

## 4.2 Riemannian metric and exponential map

A *partition of unity* on an orbifold  $X$  is a collection  $\{\rho_i : i \in \Lambda\}$  of  $C^\infty$  functions on  $X$  such that

- The collection of supports  $\{supp \rho_i : i \in \Lambda\}$  is locally finite.

- $\sum_{i \in \Lambda} \rho_i(p) = 1$  for all  $p \in X$ , and  $\rho_i(p) \geq 0$  for all  $p \in X$  and  $i \in \Lambda$ .

**Lemma 4.2.1:** *Any cover of  $X$  consisting of uniformized open subsets admits a partition of unity on  $X$  subordinate to it.*

**Proof:** Let  $\{U_i : i \in \Lambda\}$  be a locally finite refinement of the cover. By the paracompactness of  $X$ , such a refinement exists. Each  $U_i$  is also a uniformized open set, say by  $(V_i, G_i, \pi_i)$ , so that we can take a  $G_i$ -invariant, non-negative smooth function  $\tilde{f}_i$  on  $V_i$  such that  $\tilde{f}_i$  can be extended over  $X$  by zeros and  $\{\text{supp } \tilde{f}_i : i \in \Lambda\}$  also covers  $X$ , i.e.,  $\tilde{f}(p) = \sum_{i \in \Lambda} \tilde{f}_i(p) \neq 0$  for all  $p \in X$ . We define  $\rho_i = \tilde{f}_i / \tilde{f}$ . Then  $\{\rho_i : i \in \Lambda\}$  is the required partition of unity.  $\square$

**Definition 4.2.2:** *A Riemannian metric on an orbifold  $X$  is a positive, symmetric  $(0, 2)$  tensor field on  $X$ .*

Let  $U$  be a uniformized open subset of  $X$ , uniformized by  $(V, G, \pi)$ . Then any Riemannian metric on  $X$  induces a compatible  $G$ -invariant metric on  $V$ .

**Lemma 4.2.3:** *The space of all Riemannian metrics on  $X$  is a non-empty cone in the vector space of  $C^\infty$  sections of the orbifold bundle of the symmetric square of  $T^*X$ .*

**Proof:** The cone structure is obvious, we only need to show the existence. But this follows easily from the existence of partitions of unity on  $X$ .  $\square$

Let  $p \in X$  and  $(V_p, G_p, \pi_p)$  be a local chart at  $p$ . Let  $G^p$  be the isotropy subgroup of  $G_p$  at  $p$  (which is the germ of the groups  $G_p$ ). Since  $G_p$  takes geodesics in  $V_p$  to geodesics, there is a convex geodesic ball  $B_p(r)$  of radius  $r$  at  $p$  which is invariant under  $G^p$  and  $(B_p(r), G^p, \pi_p)$  is a uniformizing system of  $\pi_p(B_p(r))$  in  $\pi_p(V_p)$ . Via the exponential map, we can think of  $B_p(r)$  as a ball of radius  $r$  in  $\mathbf{R}^n$  and  $G^p$  acts as a subgroup of  $O(n)$ . We call  $(B_p(r), G^p, \pi_p)$  a *geodesic chart of radius  $r$  at  $p$*  and  $\pi_p(B_p(r))$  a *geodesic neighborhood of radius  $r$* . An open subset  $U$  of a geodesic neighborhood is called *star-shaped* if the induced uniformizing system from the geodesic chart is a star-shaped domain with respect to the origin.

**Lemma 4.2.4:** *Let  $U_p$  be a geodesic neighborhood of radius  $r$  at  $p$  and  $(V_i, G_i, \pi_i)$ ,  $i = 1, 2$ , be two uniformizing systems of  $U_p$  defining the same germ at  $p$ ; then  $(V_i, G_i, \pi_i)$ ,  $i = 1, 2$ , are isomorphic. As a consequence, for any uniformized open set  $U$  in  $X$ , uniformized by  $(V, G, \pi)$ , if a geodesic neighborhood  $U_p$  is contained in  $U$ , then there is an injection from the corresponding geodesic chart into  $(V, G, \pi)$ . The same conclusion also holds for any star-shaped open subset in a geodesic neighborhood.*

**Proof:** The uniformizing system  $(V_i, G_i, \pi_i)$  is isomorphic to the geodesic chart  $(B_p(r), G_{p,i}, \pi_{p,i})$ ,  $i = 1, 2$ . Since the two uniformizing systems define the same germ at  $p$ ,  $G_{p,1} = G_{p,2}$  and  $\pi_{p,1} = \pi_{p,2}$ , hence  $(V_i, G_i, \pi_i)$ ,  $i = 1, 2$ , are isomorphic.  $\square$

**Definition 4.2.5:** *A  $C^\infty$  path  $\tilde{\gamma} : I \rightarrow X$  in a Riemannian orbifold  $X$  is called a parametrized geodesic if for each  $t \in I$ ,  $\tilde{\gamma}_t$  is a geodesic in some chart  $(V_{\gamma(t)}, G_{\gamma(t)}, \pi_{\gamma(t)})$  at  $\gamma(t)$ . The image  $\gamma(I)$  is called a geodesic in  $X$ .*

Since on a smooth manifold a geodesic is uniquely determined by the tangent vector at one point on it, it is easily seen that the parametrized geodesic  $\tilde{\gamma}$  is in 1 : 1 correspondence with the underlying continuous map  $\gamma$ .

The length of a parametrized geodesic  $\tilde{\gamma}$  is defined to be

$$\text{length}(\tilde{\gamma}) = \int_I |\dot{\gamma}(t)| dt,$$

where  $\dot{\gamma}(t)$  is the tangent vector of  $\tilde{\gamma}$  at  $t \in I$ , as a vector in the fiber of the tangent bundle  $TX$  at  $\gamma(t)$ . It is easily seen that the norm  $|\dot{\gamma}(t)|$  is constant in  $t$ .

**Lemma 4.2.6:** *Let  $p \in X$  be any point,  $TX_p = \mathbf{R}^n/G_p$  ( $n = \dim X$ ) be the fiber of the tangent bundle  $TX$  at  $p$ . For any vector  $v \neq 0$  in  $TX_p$ , there is a unique maximal parametrized geodesic  $\tilde{\gamma} : (-\epsilon_1, \epsilon_2) \rightarrow X$  such that  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ . When  $X$  is compact,  $\epsilon_1 = \epsilon_2 = \infty$  for any  $p \in X$ .*

**Definition 4.2.7:**  *$X$  is called geodesically complete if  $\epsilon_1 = \epsilon_2 = \infty$  for any  $p \in X$ .*

**Proof:** Parametrized geodesics satisfying  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$  exist locally around  $p$ . So we can take the union of all parametrized geodesics satisfying the said conditions to get the maximal one. We only need to show that a parametrized geodesic  $\tilde{\gamma} : (a, b) \rightarrow X$  is extendable in the following sense: if there is a sequence  $t_i \in (a, b)$  such that  $t_i \rightarrow b$  and  $\lim \gamma(t_i)$  exists in  $X$ , then there is a parametrized geodesic  $\tilde{\gamma}' : (a, b') \rightarrow X$ ,  $b' > b$ , such that  $\tilde{\gamma} = \tilde{\gamma}'$  on  $(a, b)$ . This is seen as follows: Let  $q = \lim \gamma(t_i)$ . We take a geodesic chart  $(V_q, G_q, \pi_q)$  at  $q$ . For large  $i$ ,  $\gamma(t_i)$  lies in  $\pi_q(V_q)$ . Since  $V_q$  is geodesically convex, all of  $\gamma(t)$  lies in  $\pi_q(V_q)$  when  $t \rightarrow b$  and  $\lim \gamma(t) = q$  as  $t \rightarrow b$ . The extendability follows from local existence.  $\square$

Now we are ready to define the exponential map. We assume that  $X$  is geodesically complete. We define the exponential map  $\exp : TX \rightarrow X$  by

$$\exp(p, v) = \gamma_{(p, v)}(1),$$

where  $\tilde{\gamma}_{(p, v)}$  is the unique maximal parametrized geodesic satisfying  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ .

**Proposition 4.2.8:** *The exponential map  $\exp$  is continuous, and there exists a canonically defined  $C^\infty$  map, denoted by  $\text{Exp}$ , as a germ of  $C^\infty$  liftings of  $\exp$ .*

**Remark 4.2.9:** *We will introduce a notion of good map and compatible system in subsection 4.3. The proof here also shows that  $\text{Exp}$  is a good map with a unique isomorphism class of compatible systems.*

**Proof:** We first prove that  $\exp$  is continuous. Suppose  $(p_i, v_i) \in TX$  converges to  $(p, v)$ . Let  $I$  be the subset of  $(-\infty, \infty)$  consisting of  $t$  such that  $\lim \gamma_{(p_i, v_i)}(t)$  exists and equals  $\gamma_{(p, v)}(t)$ . Then an open neighborhood of 0 is in  $I$  and  $I$  is both open and closed. So  $I = (-\infty, \infty)$  and  $\gamma_{(p, v)}(1) = \lim \gamma_{(p_i, v_i)}(1)$ . Hence  $\exp$  is continuous.

The germ of  $C^\infty$  liftings  $\text{Exp}$  is characterized as follows: Each geodesic chart of  $TX$  at  $(p, v)$  is a product of a geodesic chart  $B_p$  of  $X$  at  $p$  with a ball  $B_v$  in  $\mathbf{R}^n$  centered at a preimage of  $v$  in  $\mathbf{R}^n$ . The group action is given by the subgroup  $G_{p, v}$  of  $G_p$  which fixes  $v$ . We denote such a chart of  $TX$  at  $(p, v)$  by  $(V_{p, v}, G_{p, v}, \pi_{p, v})$ . Let  $q = \exp(p, v)$ . Then any local lifting  $\tilde{f} : (V_{p, v}, G_{p, v}, \pi_{p, v}) \rightarrow (V_q, G_q, \pi_q)$  compatible with  $\text{Exp}$  has the property that the restriction of  $\tilde{f}$  to the intersection of each 1-dim linear subspace of  $\mathbf{R}^n$  with  $B_v$  (we will call these intersections *lines* in  $V_{p, v}$ ) is a parametrized geodesic in  $V_q$ .

Before we construct such local liftings of  $\exp$ , we prove that they are unique up to isomorphism. In other words, let  $\tilde{f}_i$ ,  $i = 1, 2$ , be two local liftings of  $\exp$  compatible with  $\text{Exp}$  from  $(V_{p, v}, G_{p, v}, \pi_{p, v})$  to  $(V_q, G_q, \pi_q)$ ; then there is an automorphism  $(\psi, \tau)$  (which is actually unique) of  $(V_q, G_q, \pi_q)$  such that  $\psi \circ \tilde{f}_1 = \tilde{f}_2$ . This is seen as follows: For each line  $l$  in  $V_{p, v}$ ,  $\tilde{f}_1(l)$  and  $\tilde{f}_2(l)$  are two parametrized geodesics in  $(V_q, G_q, \pi_q)$  which are mapped to the same geodesic in  $X$  under  $\pi_q$ . So there is an element  $g(l) \in G_q$  such that  $g(l) \circ \tilde{f}_1(l) = \tilde{f}_2(l)$ . When  $\tilde{f}_1(l)$  and  $\tilde{f}_2(l)$  do not lie entirely in the singular set in  $V_q$ ,  $g(l)$  is unique. This is true when  $l$  is not a singular line in  $V_{p, v}$ . The set of singular lines in  $V_{p, v}$  is of codimension at least two in the space of all lines in  $V_{p, v}$ , so that there is

an element  $g \in G_q$  such that  $g \circ \tilde{f}_1(l) = \tilde{f}_2(l)$  for any non-singular line  $l$ . By continuity,  $g \circ \tilde{f}_1 = \tilde{f}_2$  on  $V_{p,v}$ , which proves the claim.

Finally, we show that these local liftings do exist. we proceed as follows: It suffices to show that for each  $p_0 \in X$ , there is a geodesic chart  $B_{p_0}$  such that for any  $p \in B_{p_0}$ , these local liftings of  $\exp$  exist in a neighborhood of  $(p, v)$  for any  $v \in TX_p$ . This is certainly true for those  $v$ 's with small norms. We will show that for any  $r > 0$  there is a  $\epsilon_r > 0$  such that if it is true for any  $(p, v)$  where  $p \in B_{p_0}$  and  $|v| < r$ , then it is true for any  $(p, v)$  where  $|v| < r + \epsilon_r$ . This goes as follows: For any  $(p, v)$ ,  $|v| = r$ , let  $q = \exp(p, v)$ , and let  $(V_{p,v}, G_{p,v}, \pi_{p,v})$  be a geodesic chart at  $(p, v)$  and  $(V_q, G_q, \pi_q)$  be any chart at  $q$  such that  $\exp \circ \pi_{p,v}(V_{p,v}) \subset \pi_q(V_q)$ . For any  $(p', v') \in V_{p,v}$  such that  $|v'| < r$ , the local liftings defined in a local chart at  $(p', v')$  will induce local liftings via injections into  $(V_{p,v}, G_{p,v}, \pi_{p,v})$  on the images of injections in  $V_{p,v}$ . By the same reason as stated in the previous paragraph, these induced local liftings fit together to give a local lifting defined on the subset  $V$  of  $V_{p,v}$  such that  $\pi_{p,v}(V)$  consists of those  $(p', v')$  with  $|v'| < r$ . We simply extend this lifting over the whole  $V_{p,v}$  by parametrized geodesics. Now the existence of  $\epsilon_r$  for each  $r$  follows from the precompactness of the set  $\{(p, v) : p \in B_{p_0}, |v| = r\}$  in  $TX$  so that for each such a  $(p, v)$ , there is a chart at  $\exp(p, v)$  such that a geodesic ball of radius  $2\epsilon_r$  centered at a preimage of  $\exp(p, v)$  is contained in the chart, for some  $\epsilon_r > 0$ . This proves the existence of  $\exp$ .  $\square$

**Corollary 4.2.10:** *Suppose  $X$  is geodesically complete. If for  $p, q \in X$ , there is a parametrized geodesic connecting  $p$  and  $q$ , then there is a parametrized geodesic of minimal length connecting  $p$  and  $q$ .*

**Proof:** The assumption that there is a parametrized geodesic connecting  $p$  and  $q$  means that there is a  $v \in TX_p$  such that  $\exp(p, v) = q$ . Let  $v_0$  be in  $TX_p$  such that  $|v_0| = \inf |v|$  where  $\inf$  is taken over all  $v \in TX_p$  such that  $\exp(p, v) = q$ , then by continuity of  $\exp$ ,  $\exp(p, v_0) = q$  and the parametrized geodesic  $\tilde{\gamma}_{p,v_0}$  is of minimal length.  $\square$

**Definition 4.2.11:** *A parametrized geodesic connecting  $p$  and  $q$  which is of minimal length is called a minimal geodesic. An open subset  $U$  of  $X$  is called geodesically convex if for any  $p, q \in U$ , every minimal geodesic connecting  $p$  and  $q$  lies entirely in  $U$ . It is easily seen that a geodesic neighborhood of sufficiently small radius is geodesically convex, and the intersection of two geodesically convex sets is geodesically convex.*

### 4.3 Differential geometry of orbifold vector bundle

In this subsection, we discuss the Chern-Weil theory for orbifold bundles. Because a general orbifold vector bundle may not have any local section, the usual constructions of connection and curvature do not work for such an orbifold vector bundle. We call an orbifold vector bundle *good* if the local group of the base and total space have the same kernel. Equivalently, an orbifold vector bundle  $E \rightarrow X$  is good iff  $E_{red} \rightarrow X_{red}$  is an orbifold vector bundle of  $X_{red}$ . In this case, the local sections of  $E$  are in one-to-one correspondence to local sections of  $E_{red}$ . The differential geometry of  $E$  is identical to the differential geometry of  $E_{red}$ . Hence, we can assume that  $X$  is reduced for this purpose. Examples of good orbifold vector bundles include the tangent and cotangent bundles as well as their tensor and exterior products. Throughout this subsection, we assume that our orbifold vector bundle is good.

Let  $U$  be a connected topological space uniformized by  $(V, G, \pi)$ , with an orbifold bundle of rank  $k$ ,  $pr : E \rightarrow U$  uniformized by  $(V \times \mathbf{R}^k, G, \tilde{\pi})$ . We consider  $G$ -equivariant connections  $\nabla$  on  $V \times \mathbf{R}^k$ , i.e., for any smooth section  $u$  and vector field  $v$  on  $V$  and any element  $g \in G$ , we have  $g(\nabla_v u) = \nabla_{gv} gu$ . Let  $(V_i \times \mathbf{R}^k, G_i, \tilde{\pi}_i)$ ,  $i = 1, 2$ , be two isomorphic uniformizing systems of

$pr : E \rightarrow U$ , with  $G_i$ -equivariant connections  $\nabla_i$  on it. We say  $\nabla_1$  and  $\nabla_2$  are *isomorphic* if for an isomorphism  $\psi$  from  $(V_1 \times \mathbf{R}^k, G_1, \tilde{\pi}_1)$  to  $(V_2 \times \mathbf{R}^k, G_2, \tilde{\pi}_2)$ , the equation  $\psi^* \nabla_2 = \nabla_1$  holds. Then one can easily check that given a connected open subset  $U'$  of  $U$ , each isomorphism class of  $\nabla$  over  $U$  induces a unique isomorphism class of connections  $\nabla'$  over  $U'$ . One can define the *germ* at point  $p$  in the sense that  $\nabla_1$  and  $\nabla_2$  are *equivalent* at  $p$  if they induce isomorphic connections over a neighborhood of  $p$ .

**Definition 4.3.1:** Let  $pr : E \rightarrow X$  be an orbifold bundle with an orbifold bundle structure  $\mathcal{V} = \{(V_p \times \mathbf{R}^k, G_p, \tilde{\pi}_p) : p \in X\}$ . A connection  $\nabla$  on  $pr : E \rightarrow X$  is a collection of connections  $\{\nabla_p : p \in X\}$ , with each  $\nabla_p$  being a  $G_p$ -equivariant connection on  $(V_p \times \mathbf{R}^k, G_p, \tilde{\pi}_p)$  such that for any  $q \in U_p = \pi_p(V_p)$ ,  $\nabla_p$  and  $\nabla_q$  are equivalent at  $q$ . Two connections  $\nabla_i$  in the reference of orbifold bundle structures  $\mathcal{V}_i$ ,  $i = 1, 2$ , are equivalent if they induce isomorphic connections over a neighborhood of each point  $p \in X$ . We will call each equivalence class a connection on  $pr : E \rightarrow X$ . Note that if  $u$  is a  $C^\infty$  section of  $E$  and  $v$  is a  $C^\infty$  section of  $TX$ , then  $\nabla_v u$  is a  $C^\infty$  section of  $E$ .

Given two connections  $\nabla_1$  and  $\nabla_2$  on  $pr : E \rightarrow X$ , say in the reference to orbifold bundle structures  $\mathcal{V}_1$  and  $\mathcal{V}_2$ . For each  $p \in X$ , there is a uniformized neighborhood  $U_p$  such that  $\mathcal{V}_1$  and  $\mathcal{V}_2$  induce isomorphic uniformizing systems of  $E_{U_p} = pr^{-1}(U_p)$  over  $U_p$ , say  $(V_p \times \mathbf{R}^k, G_p, \tilde{\pi}_p)$ .  $\nabla_1$  and  $\nabla_2$  then induce two  $G_p$ -equivariant connections  $\nabla_{1,p}$  and  $\nabla_{2,p}$  on  $(V_p \times \mathbf{R}^k, G_p, \tilde{\pi}_p)$ , whose difference is a  $G_p$ -equivariant,  $End(\mathbf{R}^k)$ -valued smooth 1-form on  $V_p$ . From this consideration it is easily seen that the difference of  $\nabla_1$  and  $\nabla_2$  is a  $C^\infty$  section of the orbifold bundle  $T^*X \otimes End(E)$ . On the other hand, a connection on  $pr : E \rightarrow X$  added to a  $C^\infty$  section of  $T^*X \otimes End(E)$  gives rise to another connection on  $E$ .

**Lemma 4.3.2:** The set of all connections on  $pr : E \rightarrow X$  forms a non-empty affine space modelled on the space of  $C^\infty$  sections of the orbifold bundle  $T^*X \otimes End(E)$ .

**Proof:** We only need to show that the set of all connections is non-empty. Suppose  $X$  is covered by a collection of uniformized open sets  $\mathcal{U} = \{U_i : i \in \Lambda\}$ , each  $U_i$  uniformized by  $(V_i, G_i, \pi_i)$ , and let  $E_i$  be a smooth vector bundle over  $V_i$  which uniformizes  $pr^{-1}(U_i)$ . Moreover we assume there is a  $C^\infty$  partition of unity  $\{\rho_i : i \in \Lambda\}$  subordinate to the cover  $\mathcal{U}$ . Let  $\tilde{\rho}_i$  be the corresponding  $G_i$ -invariant function on  $V_i$ . Let  $\nabla_i$  be a  $G_i$ -equivariant connection on  $E_i$ . Now suppose  $\mathcal{V} = \{(V_p \times \mathbf{R}^k, G_p, \tilde{\pi}_p) : p \in X\}$  is an orbifold bundle structure of  $pr : E \rightarrow X$ ; we will construct a connection  $\nabla$  on  $E$ , that is, a collection of connections  $\nabla_p$  on  $V_p \times \mathbf{R}^k$  which is  $G_p$ -equivariant and for any  $q \in \pi_p(V_p)$ ,  $\nabla_p$  and  $\nabla_q$  are equivalent at  $q$ . We first define the connection  $\nabla_p$ . This amounts for any smooth section  $u$  of  $V_p \times \mathbf{R}^k$  and any vector field  $v$  on  $V_p$ , to define  $(\nabla_p)_v u$ , at any point  $\tilde{x} \in V_p$ . Let  $x = \pi_p(\tilde{x})$ . We take a small geodesic neighborhood  $U_x$  of  $x$  such that if  $x$  is in  $U_i$  for any  $i \in \Lambda$ ,  $U_x \subset U_i$ , and if  $U_x \cap U_j \neq \emptyset$  but  $U_x$  is not contained in  $U_j$  for some  $j \in \Lambda$ , then  $\overline{U_x} \cap \overline{U_j} \cap \text{supp } \rho_j = \emptyset$ . Let  $(V_x \times \mathbf{R}^k, G_x, \tilde{\pi}_x)$  be a uniformizing system of  $pr^{-1}(U_x)$  and  $\psi : (V_x \times \mathbf{R}^k, G_x, \tilde{\pi}_x) \rightarrow (V_p \times \mathbf{R}^k, G_p, \tilde{\pi}_p)$  an injection covering  $\tilde{x}$ . For any inclusion  $U_x \subset U_i$ , we pick an injection  $\psi_i : (V_x \times \mathbf{R}^k, G_x, \tilde{\pi}_x) \rightarrow E_i$ . We define

$$(\nabla_p)_v u|_{\tilde{x}} = [\psi_* \sum_{i \in \Lambda} (\psi_i^{-1})^* (\tilde{\rho}_i (\nabla_i)_{(\psi_i)_*(\psi^{-1})_*(v)} (\psi_i)_*(\psi^{-1})_*(u))]_{\tilde{x}}.$$

One can verify that  $(\nabla_p)_v u|_{\tilde{x}}$  is independent of the choices of  $U_x$ ,  $\psi$  and  $\psi_i$ , and is smooth in  $\tilde{x}$ , so that we have a well-defined connection on  $V_p \times \mathbf{R}^k$ , and such defined  $\nabla_p$  is also  $G_p$ -equivariant, and for any  $q \in \pi_p(V_p)$ ,  $\nabla_p$  and  $\nabla_q$  are equivalent at  $q$ . Hence the lemma.  $\square$

**Remark 4.3.3:** One can put a Riemannian metric on an orbifold bundle, or a Hermitian metric on a complex orbifold bundle, and define the corresponding notion of  $O(k)$ -connection or  $U(k)$ -

connection, and prove that the set of  $O(k)$ -connections or  $U(k)$ -connections is a non-empty affine space.

For each connection  $\nabla$  on  $pr : E \rightarrow X$ , we can define the curvature  $F(\nabla)$  of  $\nabla$  as a  $C^\infty$  section of the orbifold bundle  $\Lambda^2(X) \otimes \text{End}(E)$ . This is done as follows: let  $\nabla$  be a connection for orbifold bundle structure  $\mathcal{V} = \{(V_p \times \mathbf{R}^k, G_p, \tilde{\pi}_p) : p \in X\}$ , by  $\{\nabla_p : p \in X\}$ . Then the curvature  $F(\nabla_p)$  of  $\nabla_p$  is a  $G_p$ -equivariant smooth 2-form on  $V_p$  with values in  $\text{End}(\mathbf{R}^k)$ . The collection of  $\{F(\nabla_p) : p \in X\}$  defines a  $C^\infty$  section of  $\Lambda^2(X) \otimes \text{End}(E)$ . We denote it by  $F(\nabla)$ . The Chern-Weil construction applied to each  $F(\nabla_p)$  yields the following

**Proposition 4.3.4:** *To each invariant polynomial, there is a characteristic class in the following sense. To each (isomorphism class of) orbifold bundle  $pr : E \rightarrow X$ , there is an associated cohomology class in the deRham cohomology group  $H_{de}^*(X)$  of  $X$  as follows: each connection on  $E$  is assigned a closed differential form on  $X$ , and for two different connections on  $E$ , the associated closed forms differ by an exact form on  $X$ . As a consequence, the Chern classes, the Pontrjagin classes and the Euler class are all well-defined in the category of orbibundles.*

**Remark 4.3.5:** *The same thing holds for the more general notion of orbifold bundles.*

**Example 4.3.6a:** *Let  $(\Sigma, j, z_1, \dots, z_k)$  be a smooth complex curve with a set of finitely many distinct points  $(z_1, \dots, z_k)$ . An orbifold structure on  $(\Sigma, j)$  can be given by a collection of uniformizing systems  $\tilde{D}_i$  at each  $z_i$ , i.e, a collection of disk neighborhoods  $D_i$  of  $z_i$  and branched covering maps  $\tilde{D}_i \rightarrow D_i$  by  $z \rightarrow z^{m_i}$ . The numbers  $m_i$  are called the multiplicities at  $z_i$ . The orbifold  $(\Sigma, j)$  thus defined is an example of a complex orbifold. Its cotangent bundle, or the canonical bundle, denoted by  $K_\Sigma$ , is an example of a holomorphic line orbifold bundle. The first Chern number  $c_1(K_\Sigma)([\Sigma])$  is equal to*

$$2g_\Sigma - 2 + \sum_{i=1}^k \left(1 - \frac{1}{m_i}\right),$$

where  $g_\Sigma$  is the genus of  $\Sigma$ . This example shows that the characteristic classes of an orbifold bundle are rational classes in general.

**Example 4.3.6b:** *Let  $\tilde{E}$  be a holomorphic line bundle over a smooth complex curve  $\tilde{\Sigma}$ , with a finite group  $G$  acting holomorphically on it. Let  $\Sigma = \tilde{\Sigma}/G$  and  $\pi : \tilde{\Sigma} \rightarrow \Sigma$  be the natural branched covering map. Then  $E = \tilde{E}/G$  is a holomorphic line orbifold bundle over  $\Sigma$ . We have the following relation amongst the Chern classes of  $\tilde{E}$  and  $E$ :  $\pi^*(c_1(E)) = c_1(\tilde{E})$ , or equivalently, in terms of Chern numbers:  $c_1(E)([\Sigma]) = \frac{1}{|G|}c_1(\tilde{E})([\tilde{\Sigma}])$ .*

#### 4.4 Pull-back orbifold bundles and good maps

Let  $pr : E \rightarrow Y$  be a vector bundle over a topological space  $Y$ . Then for any continuous map  $f : X \rightarrow Y$  from a topological space  $X$ , the pull-back vector bundle  $f^*E$  over  $X$  is well-defined. However, this is no longer the case for orbifold bundles. Let  $pr : E \rightarrow X'$  be an orbifold bundle over  $X'$ , and  $\tilde{f} : X \rightarrow X'$  a  $C^\infty$  map. By a *pull-back orbifold bundle of  $E$  over  $X$  via  $\tilde{f}$*  we mean an orbifold bundle  $\pi : E^* \rightarrow X$  together with a  $C^\infty$  map  $\tilde{f} : E^* \rightarrow E$  such that each local lifting of  $\tilde{f}$  is an isomorphism restricted to each fiber, and  $\tilde{f}$  covers the  $C^\infty$  map  $\tilde{f}$  between the bases.

Let  $\tilde{f} : X \rightarrow X'$  be a  $C^\infty$  map between orbifolds  $X$  and  $X'$  whose underlying continuous map is denoted by  $f$ . Suppose there is a compatible cover  $\mathcal{U}$  of  $X$ , and a collection of open subsets  $\mathcal{U}'$  of  $X'$  satisfying (4.1.1a–c) and the following condition: There is a 1:1 correspondence between elements of  $\mathcal{U}$  and  $\mathcal{U}'$ , say  $U \leftrightarrow U'$ , such that  $f(U) \subset U'$ , and an inclusion  $U_2 \subset U_1$  implies an inclusion  $U'_2 \subset U'_1$ . Moreover, there is a collection of local  $C^\infty$  liftings  $\{\tilde{f}_{UU'}\}$  of  $f$ , where  $\tilde{f}_{UU'} : (V, G, \pi) \rightarrow (V', G', \pi')$



satisfies the following condition: each injection  $i : (V_2, G_2, \pi_2) \rightarrow (V_1, G_1, \pi_1)$  is assigned with an injection  $\lambda(i) : (V'_2, G'_2, \pi'_2) \rightarrow (V'_1, G'_1, \pi'_1)$  such that  $\tilde{f}_{U_1 U'_1} \circ i = \lambda(i) \circ \tilde{f}_{U_2 U'_2}$ , and for any composition of injections  $j \circ i$ , the following compatibility condition holds:

$$(4.4.1) \quad \lambda(j \circ i) = \lambda(j) \circ \lambda(i).$$

Observe that when the injection  $i : (V, G, \pi) \rightarrow (V, G, \pi)$  is an automorphism of  $(V, G, \pi)$ , the assignment of  $\lambda(i) : (V', G', \pi') \rightarrow (V', G', \pi')$  to  $i$  satisfying (4.4.1) is equivalent to a homomorphism  $\lambda_{UU'} : G \rightarrow G'$ . We call  $\lambda_{UU'} : G \rightarrow G'$  the *group homomorphism* of  $\{\tilde{f}_{UU'}, \lambda\}$  on  $U$ . Such a collection of maps clearly defines a  $C^\infty$  lifting of the continuous map  $f$ . If it is in the same germ of  $\tilde{f}$ , we call  $\{\tilde{f}_{UU'}, \lambda\}$  a *compatible system* of  $\tilde{f}$ .

**Definition 4.4.1:** A  $C^\infty$  map is called *good* if it admits a compatible system.

**Example 4.4.2a:** Not every  $C^\infty$  map is good, as shown in the following example: consider an effective linear representation of a finite group  $(\mathbf{R}^n, G)$ . Let  $H^g$  be the linear subspace of fixed points of an element  $1 \neq g \in G$ . Then the centralizer  $C_G(g)$  of  $g$  in  $G$  acts on  $H^g$ , and  $(H^g, C_G(g)/K_g)$  is an effective linear representation, where  $K_g \subset C_G(g)$  is the isotropy subgroup of  $H^g$ . Suppose  $H^g \neq \{0\}$  and there is no homomorphism  $\lambda : C_G(g)/K_g \rightarrow C_G(g)$  such that  $\pi \circ \lambda = \text{id}$ , where  $\pi : C_G(g) \rightarrow C_G(g)/K_g$  is the projection; then the inclusion  $H^g \rightarrow \mathbf{R}^n$  is a  $C^\infty$  map but not a good one.

**Example 4.4.2b:** There can be essentially different compatible systems of the same  $C^\infty$  map, as shown in the following example: Let  $X = \mathbf{C} \times \mathbf{C}/G$  where  $G = \mathbf{Z}_2 \oplus \mathbf{Z}_2$  acts on  $\mathbf{C} \times \mathbf{C}$  in the standard way. For the  $C^1$  map  $\tilde{f} : (\mathbf{C}, \mathbf{Z}_2) \rightarrow (\mathbf{C} \times \mathbf{C}, G)$  defined by the inclusion of  $\mathbf{C} \times \{0\}$ , there are two compatible systems  $(\tilde{f}, \lambda_i) : (\mathbf{C}, \mathbf{Z}_2) \rightarrow (\mathbf{C} \times \mathbf{C}, G)$ ,  $i = 1, 2$ , for  $\lambda_1(1) = (1, 0)$  and  $\lambda_2(1) = (1, 1)$ , which are apparently different.

**Lemma 4.4.3:** Let  $pr : E \rightarrow X'$  be an orbifold bundle over  $X'$ . For any  $C^\infty$  compatible system  $\xi = \{\tilde{f}_{UU'}, \lambda\}$  of a good  $C^\infty$  map  $\tilde{f} : X \rightarrow X'$ , there is a canonically constructed pull-back orbifold bundle of  $E$  via  $\tilde{f}$ : an orbifold bundle  $pr : E_\xi^* \rightarrow X$  together with a  $C^\infty$  map  $f_\xi : E_\xi^* \rightarrow E$  covering  $\tilde{f}$ . Suppose that  $E$  is good.  $E_\xi^*$  is obviously good as well. Let  $c$  be a universal characteristic class defined by the Chern-Weil construction; then  $\tilde{f}^*(c(E)) = c(E_\xi^*)$ .

**Proof:** Without loss of generality, we assume that  $E_{U'}$  over  $U'$  is uniformized by  $(V' \times \mathbf{R}^k, G', \tilde{\pi}')$ . Then we have a collection of pull-back bundles  $\tilde{f}_{U'}^*(V' \times \mathbf{R}^k)$  over  $V$  which has the form of  $V \times \mathbf{R}^k$ . Let  $\{g'\}$  be a collection of transition maps of  $E$  with respect to  $\mathcal{U}'$ , we define a set of transition maps  $\{g\}$  on  $X$  with respect to the cover  $\mathcal{U}$  by pull-backs, i.e., we set  $g_i = g'_{\lambda(i)} \circ \tilde{f}_{U_2 U'_2}$  for any injection  $i : (V_2, G_2, \pi_2) \rightarrow (V_1, G_1, \pi_1)$ , where  $\lambda(i) : (V'_2, G'_2, \pi'_2) \rightarrow (V'_1, G'_1, \pi'_1)$  is the injection assigned to  $i$ . Then the compatibility condition (4.4.1) implies that the set of maps  $\{g\}$  satisfies the equation (4.1.2), which defines an orbifold bundle over  $X$ . We denote it by  $pr : E^* \rightarrow X$ . The existence of a  $C^\infty$  map  $\tilde{f} : E^* \rightarrow E$  is obvious from the construction. On the other hand, for any connection  $\nabla$  on  $E$ , there is a pull-back connection  $\tilde{f}^*(\nabla)$  on  $E^*$ , so that the equation  $\tilde{f}^*(c(E)) = c(E^*)$  holds for any universal characteristic class  $c$  defined by the Chern-Weil construction.  $\square$

**Definition 4.4.4:** Two compatible systems  $\xi_i$  for  $i = 1, 2$  of a  $C^\infty$  map  $\tilde{f} : X \rightarrow X'$  are *isomorphic* if for any orbifold bundle  $E$  over  $X'$ , there is an isomorphism  $\psi$  between the corresponding pull-back orbibundles  $E_i^*$  with  $\tilde{f}_i : E_i^* \rightarrow E$ ,  $i = 1, 2$ , such that  $\tilde{f}_1 = \psi \circ \tilde{f}_2$ .

For technical reasons, we need to construct some standard compatible systems from a given compatible system. We first give the definition.

**Definition 4.4.5:** A compatible system  $\{\tilde{f}_{UU'}, \lambda\}$  is called a geodesic compatible system if  $\mathcal{U}$  and  $\mathcal{U}'$  satisfy the following conditions: There is a countable dense subset  $\Lambda$  of  $X$  and for each point  $p$  in  $\Lambda$ , there is a sequence of geodesic neighborhoods  $U_{p,i}$  of radius  $r_i \searrow 0$ . Moreover,  $\mathcal{U} = \{U_{p,i} : p \in \Lambda, i \in \mathbf{Z}_+\}$ , and each element  $U'_{p,i}$  in  $\mathcal{U}'$  is a star-shaped neighborhood of  $p$  whose diameter goes to zero as  $i \rightarrow \infty$ .

The technique in the proof of the following lemma will be frequently used.

**Lemma 4.4.6:** For any compatible system  $\{\tilde{f}_{UU'}, \lambda\}$ , there is a geodesic compatible system  $\{\tilde{f}_{WW'}, \lambda_1\}$  such that for each pair  $(W, W')$ , there is  $(U, U')$  such that  $\tilde{f}_{WW'}$  is induced from  $\tilde{f}_{UU'}$  for some injections  $W \rightarrow U$  and  $W' \rightarrow U'$ , and  $\lambda_1$  is also induced from  $\lambda$  in a sense that will be clear from the proof.

Such a geodesic compatible system is called an *induced* compatible system.

**Proof:** We take a countable subset  $\mathcal{U}_0 = \{U_i : i \in \mathbf{Z}_+\}$  of  $\mathcal{U}$  which covers  $X$ . Then we construct a countable dense subset  $\Lambda$  of  $X$  which is given as the union of the following sequence of subsets:

$$\Lambda_1, \Lambda_2, \dots,$$

where  $\Lambda_k$  is at most countable and locally finite, and each point  $p$  in  $\Lambda_k$  comes with a geodesic neighborhood  $W_p$  of radius less than  $\frac{1}{k}$  such that if  $U_i \in \mathcal{U}_0$  is the first element containing  $p$ , then  $W_p \subset U_i$  and  $f(W_p)$  is contained in a geodesic neighborhood  $W'_{f(p)} \subset U'_{p,i}$ , and any point in  $X$  is contained in  $W_p$  for some point  $p$  in  $\Lambda_k$  for each  $k$ . This is done as follows: For each integer  $k \geq 1$ , for any point  $p \in X$ , we take a geodesic neighborhood  $W_p$  of radius less than  $\frac{1}{k}$  and if  $U_i \in \mathcal{U}_0$  is the first element containing  $p$ , then  $W_p \subset U_i$  and  $f(W_p)$  is contained in a geodesic neighborhood  $W'_{f(p)} \subset U'_{p,i}$ . Then we obtain a cover  $\{W_p : p \in X\}$  of  $X$ . We take a subcover of it which is locally finite, and we obtain the set  $\Lambda_k$ .

Now for each point  $p \in \Lambda \cap U_1$ , we take a sequence of geodesic neighborhoods  $\{U_{p,i} : i \in \mathbf{Z}_+\}$  of radius  $r_i \searrow 0$  with  $U_{p,1} = W_p$ , and a sequence of corresponding geodesic neighborhoods  $\{U'_{p,i} : i \in \mathbf{Z}_+\}$  of radius  $r'_i \searrow 0$  with  $U'_{p,1} = W'_{f(p)}$  and  $f(U_{p,i}) \subset U'_{p,i}$ . It is easily seen that  $\mathcal{W}_1 = \{U_{p,i} : i \in \mathbf{Z}_+, p \in \Lambda \cap U_1\}$  forms a compatible cover of  $U_1$ . However, an inclusion  $U_{q,j} \subset U_{p,i}$  may not imply an inclusion  $U'_{q,j} \subset U'_{p,i}$ . We overcome this problem by changing  $U'_{q,j}$  to  $U'_{q,j} \cap U'_{p,i}$ , which is a geodesically convex and star-shaped neighborhood of  $f(q)$ . This can be done because there are only finitely many  $U_{p,i}$ 's satisfying  $U_{q,j} \subset U_{p,i}$ . Now for each  $U_{p,i} \in \mathcal{W}_1$ , the corresponding  $U'_{p,i}$  is a geodesically convex and star-shaped neighborhood of  $f(p)$ . We let  $\mathcal{W}'_1 = \{U'_{p,i} : i \in \mathbf{Z}_+, p \in \Lambda \cap U_1\}$ . For each inclusion  $U_{p,j} \subset U_{p,i}$ , there is a canonical injection between the corresponding geodesic charts  $(V_{p,j}, G_p, \pi_{p,j}) \rightarrow (V_{p,i}, G_p, \pi_{p,i})$  which induces identity map on the tangent space at  $p$  and identity homomorphism  $G_p \rightarrow G_p$ . Similar injections exist between  $(V'_{p,i}, G_{f(p)}, \pi'_{p,i})$ 's, the geodesic uniformising systems of  $U'_{p,i}$ 's. We call these canonical injections *geodesic injections*. On the other hand, for each  $U_{p,i} \in \mathcal{W}_1$ , we fix an induced injection  $i_{p,i} : (V_{p,i}, G_p, \pi_{p,i}) \rightarrow (V_1, G_1, \pi_1)$  and an induced injection  $i'_{p,i} : (V'_{p,i}, G_{f(p)}, \pi'_{p,i}) \rightarrow (V'_1, G'_1, \pi'_1)$  such that  $f_{U_1 U'_1} \circ i_{p,i}(V_{p,i}) \subset i'_{p,i}(V'_{p,i})$  so that we can induce a lifting  $\tilde{f}_{p,i} = (i'_{p,i})^{-1} \circ f_{U_1 U'_1} \circ i_{p,i}$  from  $(V_{p,i}, G_p, \pi_{p,i})$  to  $(V'_{p,i}, G_{f(p)}, \pi'_{p,i})$ . We can further require that  $i_{p,j} = i_{p,i} \circ g_{i,j}$  and  $i'_{p,j} = i'_{p,i} \circ g'_{i,j}$  where  $g_{i,j}$  and  $g'_{i,j}$  are the geodesic injections. We call the injections  $i_{p,i}$  and  $i'_{p,i}$  *home injections*, and the open set  $U_1 \in \mathcal{U}_0$  the *home* of each element  $U_{p,i}$  in  $\mathcal{W}_1$ .

To each injection  $i : (V_{q,j}, G_q, \pi_{q,j}) \rightarrow (V_{p,i}, G_p, \pi_{p,i})$ , we need to assign an injection  $\lambda_1(i) : (V'_{q,j}, G_{f(q)}, \pi'_{q,j}) \rightarrow (V'_{p,i}, G_{f(p)}, \pi'_{p,i})$  such that  $\lambda_1(i) \circ \tilde{f}_{q,j} = \tilde{f}_{p,i} \circ i$  and the compatibility condition (4.4.1) holds for  $\lambda_1$ . First observe that for each injection  $i : (V_{q,j}, G_q, \pi_{q,j}) \rightarrow (V_{p,i}, G_p, \pi_{p,i})$ , there is a unique automorphism  $g(i) \in G_1$  of  $(V_1, G_1, \pi_1)$  such that  $i_{p,i} \circ i = g(i) \circ i_{q,j}$ . The uniqueness of  $g$  ensures  $g(j \circ i) = g(j) \circ g(i)$ . The image of  $g(i)$  under the group homomorphism  $\lambda_{U_1 U'_1} : G_1 \rightarrow G'_1$ ,

denoted by  $g'(i)$ , determines a unique injection  $i' : (V'_{q,j}, G_{f(q)}, \pi'_{q,j}) \rightarrow (V'_{p,i}, G_{f(p)}, \pi'_{p,i})$  by the equation  $i'_{p,i} \circ i' = g'(i) \circ i'_{q,j}$ . We define  $\lambda_1(i) = i'$ , which satisfies (4.4.1).

Now we move to the next stage of construction. For each point  $p \in \Lambda \cap (U_2 \setminus U_1)$ , we similarly take a sequence of geodesic neighborhoods  $U_{p,i}$  with corresponding  $U'_{p,i}$  which are geodesically convex and star-shaped. We define  $\mathcal{W}_2 = \{U_{p,i}\}$  and  $\mathcal{W}'_2 = \{U'_{p,i}\}$ . We can similarly fix a home injection for each element in  $\mathcal{W}_2$  and  $\mathcal{W}'_2$  and define local liftings  $\tilde{f}_{p,i}$  and the assignment  $\lambda_1$ , just as what we have done for the points in  $\Lambda \cap U_1$ . We call  $U_2$  the *home* of each element of  $\mathcal{W}_2$ .

There is a possibility that for an inclusion  $W_1 \subset W_2$  with different homes, here  $W_1 \in \mathcal{W}_1$  and  $W_2 \in \mathcal{W}_2$ , there may not be an inclusion  $W'_1 \subset W'_2$ . We overcome this problem again by changing  $W'_1$  to the intersection  $W'_1 \cap W'_2$  which remains to be geodesically convex and star-shaped. With this done, we take  $\mathcal{W}_{12} = \mathcal{W}_1 \cup \mathcal{W}_2$  and  $\mathcal{W}'_{12} = \mathcal{W}'_1 \cup \mathcal{W}'_2$ . It is easily seen that  $\mathcal{W}_{12}$  is a compatible cover of  $U_1 \cup U_2$ . The last thing we need to do in order to complete the construction of a geodesic compatible system on  $U_1 \cup U_2$  is to assign to each injection  $i : (V_{q,j}, G_q, \pi_{q,j}) \rightarrow (V_{p,i}, G_p, \pi_{p,i})$ , where  $U_{q,j}$  has home  $U_1$  and  $U_{p,i}$  has home  $U_2$ , an injection  $\lambda_1(i) : (V'_{q,j}, G_{f(q)}, \pi'_{q,j}) \rightarrow (V'_{p,i}, G_{f(p)}, \pi'_{p,i})$  such that  $\lambda_1(i) \circ \tilde{f}_{q,j} = \tilde{f}_{p,i} \circ i$  and (4.4.1) holds for  $\lambda_1$ . Since the set of injections  $i : (V_{q,j}, G_q, \pi_{q,j}) \rightarrow (V_{p,i}, G_p, \pi_{p,i})$  is in 1:1 correspondence with the set of injections  $i_1 : (V_{q,j_1}, G_q, \pi_{q,j_1}) \rightarrow (V_{p,i}, G_p, \pi_{p,i})$  through the geodesic injection  $g_{j,j_1}$ , it suffices to define  $\lambda_1(i)$  for those  $U_{q,j}$  with  $j$  large. This is done as follows: First observe that  $q \in U_1 \cap U_2$ , so that there is a  $U \in \mathcal{U}$  such that  $U \subset U_1 \cap U_2$  and  $U_{q,j} \subset U$  by taking large  $j$ . For each injection  $i : (V_{q,j}, G_q, \pi_{q,j}) \rightarrow (V_{p,i}, G_p, \pi_{p,i})$ , there are injections  $i_{UU_1} : (V, G, \pi) \rightarrow (V_1, G_1, \pi_1)$  and  $i_{UU_2} : (V, G, \pi) \rightarrow (V_2, G_2, \pi_2)$  such that  $i = i_{p,i}^{-1} \circ i_{UU_2} \circ i_{UU_1}^{-1} \circ i_{q,j}$ . We define  $\lambda_1(i) = (i'_{p,i})^{-1} \circ \lambda(i_{UU_2}) \circ \lambda(i_{UU_1})^{-1} \circ i'_{q,j}$ . The fact that  $\lambda$  satisfies (4.4.1) implies that such defined  $\lambda_1$  is independent of the choices of  $U$  and  $i_{UU_i}$ 's. One can also verify that  $\lambda_1$  satisfies  $\lambda_1(i) \circ \tilde{f}_{q,j} = \tilde{f}_{p,i} \circ i$  and (4.4.1). By repeating this process, we obtain the required geodesic compatible system.  $\square$

**Remark 4.4.7:** From the construction it is easily seen that given a set of finitely many compatible systems, we can assume that the corresponding induced compatible systems have common  $\mathcal{W}$  and  $\mathcal{W}'$ .

**Proposition 4.4.8:** Two compatible systems are isomorphic if and only if there are corresponding induced compatible systems  $\{\tilde{f}_{1,WW'}, \lambda_1\}$  and  $\{\tilde{f}_{2,WW'}, \lambda_2\}$ , and automorphisms  $\{\delta_{V'}\}$  of the uniformizing system  $(V', G', \pi')$  of each element  $W' \in \mathcal{W}'$ , such that  $\delta_{V'} \circ \tilde{f}_{1,WW'} = \tilde{f}_{2,WW'}$ , and for any injection  $i : (W_2, G_2, \pi_2) \rightarrow (W_1, G_1, \pi_1)$ , we have  $\lambda_2(i) = \delta_{V'} \circ \lambda_1(i) \circ (\delta_{V'})^{-1}$ .

**Proof:** First observe that the induced compatible system defines identical pull-back orbibundles. Secondly, if  $\{\tilde{f}_{1,WW'}, \lambda_1\}$ ,  $\{\tilde{f}_{2,WW'}, \lambda_2\}$ , and  $\{\delta_{V'}\}$  satisfy  $\delta_{V'} \circ \tilde{f}_{1,WW'} = \tilde{f}_{2,WW'}$ , and for any injection  $i : (W_2, G_2, \pi_2) \rightarrow (W_1, G_1, \pi_1)$ ,  $\lambda_2(i) = \delta_{V'} \circ \lambda_1(i) \circ (\delta_{V'})^{-1}$ , then for any orbifold bundle  $E$  over  $X'$ , if  $E_i^*$ ,  $i = 1, 2$ , are the corresponding pull-back orbibundles with  $\tilde{f}_i : E_i^* \rightarrow E$ , we have an isomorphism  $\delta : E_1^* \rightarrow E_2^*$  determined by  $\delta_{V'}$  by (4.1.3), which satisfies  $\tilde{f}_2 \circ \delta = \tilde{f}_1$ . Hence the proposition in one direction.

On the other hand, for any two isomorphic compatible systems, we take their induced compatible systems  $\{\tilde{f}_{1,WW'}, \lambda_1\}$  and  $\{\tilde{f}_{2,WW'}, \lambda_2\}$ , which has common  $W$  and  $W'$ . Then the isomorphism between the pull-back orbibundles gives rise to a collection of automorphisms  $\delta_{V'}$  such that  $\tilde{f}_{2,WW'} = \delta_{V'} \circ \tilde{f}_{1,WW'}$  and for any injection  $i : (W_2, G_2, \pi_2) \rightarrow (W_1, G_1, \pi_1)$ ,

$$g_{\lambda_2(i)} \circ \tilde{f}_{2,WW'} = \delta_{V'} \circ g_{\lambda_1(i)} \circ (\delta_{V'})^{-1} \circ \tilde{f}_{2,WW'},$$

where  $\{g\}$  is the set of transition maps of orbifold bundle  $E$  over  $X'$ . We take  $E = TX'$ , then  $g_i = di$  for any injection  $i$ , so we have  $\lambda_2(i) = \delta_{V'} \circ \lambda_1(i) \circ (\delta_{V'})^{-1}$ .  $\square$

**Definition 4.4.9:** It is easily seen that for each  $p \in X$ , a compatible system determines a group homomorphism  $G_p \rightarrow G_{f(p)}$ , and an isomorphism class of compatible systems determines a conjugacy class of homomorphisms. We call such a conjugacy class of group homomorphisms the group homomorphism of the said isomorphism class of compatible systems at  $p$ .

**Definition 4.4.10:** A  $C^\infty$  map  $\tilde{f} : X \rightarrow X'$  is called regular if the underlying continuous map  $f$  has the following property:  $f^{-1}(X'_{reg})$  is an open dense and connected subset of  $X$ .

**Lemma 4.4.11:** If  $\tilde{f}$  is regular, then  $\tilde{f}$  is the unique germ of  $C^\infty$  liftings of  $f$ . Moreover,  $\tilde{f}$  is good with a unique isomorphism class of compatible systems.

**Proof:** We need to show that if  $\tilde{f}_1, \tilde{f}_2 : (V, G, \pi) \rightarrow (V', G', \pi')$  are two local liftings of  $f : \pi(V) \rightarrow \pi'(V')$ , then there is an automorphism  $\psi$  of  $(V', G', \pi')$  such that  $\tilde{f}_2 = \psi \circ \tilde{f}_1$ . Note that if for  $p \in V$  such that  $f(\pi(p))$  is regular in  $\pi'(V')$ , there is a unique automorphism  $\psi_p$  of  $(V', G', \pi')$  such that  $\tilde{f}_2(p) = \psi_p \circ \tilde{f}_1(p)$ .  $\psi_p$  is locally constant in  $p$ . So the fact that  $f^{-1}(X'_{reg})$  is an open dense and connected subset of  $X$  implies that there is a unique  $\psi$  such that  $\tilde{f}_2 = \psi \circ \tilde{f}_1$ . This proves the uniqueness of  $\tilde{f}$  as the germ of  $C^\infty$  liftings of  $f$ .

In order to show that  $\tilde{f}$  is good, we need to construct a compatible system of  $\tilde{f}$ . Suppose there is a compatible cover  $\mathcal{U}$  of  $X$ , and a collection of open subsets  $\mathcal{U}'$  of  $X'$  satisfying (4.1.1a – c) and the following condition: there is a 1:1 correspondence between elements of  $\mathcal{U}$  and  $\mathcal{U}'$ , say  $U \leftrightarrow U'$ , such that  $f(U) \subset U'$  and  $U_2 \subset U_1$  implies  $U'_2 \subset U'_1$ . Moreover, there is a collection of local  $C^\infty$  liftings  $\{\tilde{f}_{UU'}\}$  of  $f$  where  $\tilde{f}_{UU'} : (V, G, \pi) \rightarrow (V', G', \pi')$  satisfy the following condition: for any injection  $i : (V_2, G_2, \pi_2) \rightarrow (V_1, G_1, \pi_1)$ , there is an injection  $\lambda(i) : (V'_2, G'_2, \pi'_2) \rightarrow (V'_1, G'_1, \pi'_1)$  such that  $\tilde{f}_{U_1 U'_1} \circ i = \lambda(i) \circ \tilde{f}_{U_2 U'_2}$ . Then the condition that  $\tilde{f}$  is regular implies that each  $\lambda(i)$  is unique to  $i$  so that (4.4.1) automatically holds for  $\lambda$ . Now we can appeal to the general construction of induced compatible systems to show that  $\tilde{f}$  is good.

For any two compatible systems of  $\tilde{f}$ , we have two induced compatible systems  $\{\tilde{f}_{1, WW'}, \lambda_1\}$  and  $\{\tilde{f}_{2, WW'}, \lambda_2\}$ . There is a collection of automorphisms  $\{\delta_{V'}\}$  of the uniformizing system  $(V', G', \pi')$  of each element  $W' \in \mathcal{W}'$ , such that  $\delta_{V'} \circ \tilde{f}_{1, WW'} = \tilde{f}_{2, WW'}$ . For any injection  $i : (W_2, G_2, \pi_2) \rightarrow (W_1, G_1, \pi_1)$ , we define  $\tau_2(i) = \delta_{V'_1} \circ \lambda_1(i) \circ (\delta_{V'_2})^{-1}$ . Then we have a compatible system  $\{\tilde{f}_{2, WW'}, \tau_2\}$  which is isomorphic to  $\{\tilde{f}_{1, WW'}, \lambda_1\}$ . Now  $\tilde{f}$  is regular implies that  $\tau_2 = \lambda_2$ . Hence the lemma.  $\square$

**Remark 4.4.12a:** Similar argument shows that the exponential map  $Exp$  is a good  $C^\infty$  map with a unique isomorphism class of compatible systems. The key point here is that each line segment in  $TX$  is mapped to a parametrized geodesic and the set of singular lines is of codimension at least two. See the construction of  $Exp$  for details.

**Remark 4.4.12b:** Here are some examples of good  $C^\infty$  maps. Let  $E$  be an orbifold bundle over  $X$ . By the definition, the projection is good. Moreover, any  $C^\infty$  section  $\tilde{s}$  of  $E$ , as a  $C^\infty$  map  $X \rightarrow E$ , is also good. If  $X$  is reduced, both of them are regular. Let  $f$  be a good map. The map  $f^*E \rightarrow E$  is again a good map.

**Remark 4.4.12c:** Here is another class of regular  $C^\infty$  maps. A  $C^\infty$  map  $\tilde{f} : X \rightarrow X'$  is called a  $C^\infty$  embedding if each local lifting  $\tilde{f}_p : (V_p, G_p, \pi_p) \rightarrow (V_{f(p)}, G_{f(p)}, \pi_{f(p)})$  is a  $\lambda_p$ -equivariant embedding for some isomorphism  $\lambda_p : G_p \rightarrow G_{f(p)}$ . If  $X'$  is reduced, it is easily seen that  $f^{-1}(\Sigma X') = \Sigma X$  so that  $\tilde{f}$  is regular. For a  $C^\infty$  embedding  $\tilde{f} : X \rightarrow X'$ , the normal bundle of  $f(X)$  in  $X'$  as an orbifold bundle is well-defined, which is isomorphic to the quotient orbifold bundle  $(TX')^*/TX$ .

**Example 4.4.13:** Let  $\mathbf{P}^1 = \{[z_0, z_1]\}$  be the 1-dimensional complex projective space. We define a  $\mathbf{Z}_2$  action on it by  $x \cdot [z_0, z_1] = [xz_0, z_1]$ . Let  $X = \mathbf{P}^1/\mathbf{Z}_2$  be the orbifold as quotient space. Similarly,

let  $\mathbf{P}^2 = \{[z_0, z_1, z_2]\}$  be the 2-dimensional complex projective space, with a  $Z_2 \oplus Z_2$  action on it, given by  $(x, y) \cdot [z_0, z_1, z_2] = [xz_0, yz_1, z_2]$ . Let  $X' = \mathbf{P}^2/(Z_2 \oplus Z_2)$  be the orbifold as quotient space. We consider two sequences of  $C^\infty$  maps (actually they are holomorphic)  $\tilde{f}_n, \tilde{g}_n : X \rightarrow X'$  defined by  $\tilde{f}_n([z_0, z_1]) = [z_0, n^{-1}z_1, z_1]$  and  $\tilde{g}_n([z_0, z_1]) = [z_0, n^{-1}z_0, z_1]$ . It is easily seen that both sequences consist of regular maps, so they are good maps. As  $n \rightarrow \infty$ , both sequences converge. Let  $\tilde{f} = \lim \tilde{f}_n$  and  $\tilde{g} = \lim \tilde{g}_n$ . Then both  $\tilde{f}$  and  $\tilde{g}$  are good maps (as we shall see), and  $\tilde{f} = \tilde{g}$  as  $C^\infty$  maps, but with different isomorphism classes of compatible systems. In fact, the group homomorphism of  $\tilde{f}$  which is from  $\mathbf{Z}_2$  to  $\mathbf{Z}_2 \oplus \mathbf{Z}_2$  is given by  $x \rightarrow (x, 1)$  at  $[0, 1]$ , and  $x \rightarrow (x, x)$  at  $[1, 0]$ . For  $\tilde{g}$ , it is given by  $x \rightarrow (x, x)$  at  $[0, 1]$  and  $x \rightarrow (1, x)$  at  $[1, 0]$ .

□

The operation of composition is well-defined for good maps, as shown in the following lemma.

**Lemma 4.4.14:** *Let  $\tilde{f}, \tilde{g}$  be two good  $C^\infty$  maps, then the composition  $\tilde{g} \circ \tilde{f}$  is also a good  $C^\infty$  map, and any isomorphism class of compatible systems of  $\tilde{f}$  and  $\tilde{g}$  determines a unique isomorphism class of compatible systems for the composition  $\tilde{g} \circ \tilde{f}$ .*

**Proof:** We need to show that the composition of two good maps is again a good map. The rest of the lemma follows easily from Definition 4.4.4.

Without loss of generality, we assume that the compatible systems of  $\tilde{f}$  and  $\tilde{g}$  are geodesic, given by  $\{\tilde{f}_{U\mathcal{U}'}, \tau\}$  and  $\{\tilde{g}_{U_1\mathcal{U}'_1}, \lambda\}$ , where  $\mathcal{U} = \{U_{p,i} : p \in \Lambda, i \in \mathbf{Z}_+\}$  and  $\mathcal{U}_1 = \{U_{p,i} : p \in \Lambda', i \in \mathbf{Z}_+\}$ . In the construction of  $\{\tilde{g}_{U_1\mathcal{U}'_1}, \lambda\}$  we can add  $f(\Lambda)$  into  $\Lambda'$  trivially so we can assume that  $f(\Lambda) \subset \Lambda'$ . Moreover, we can assume that each  $U'_{p,i} \in \mathcal{U}'$  is contained in a  $U_{f(p),j} \in \mathcal{U}_1$ . However, there is one problem: an inclusion between elements in  $\mathcal{U}'$  may not imply a corresponding inclusion between elements in  $\mathcal{U}_1$  and therefore an inclusion between the corresponding elements in  $\mathcal{U}'_1$ . We can overcome this problem by changing the elements in  $\mathcal{U}'_1$  to suitable finite intersections which are again geodesically convex and star-shaped. Let  $\tilde{h} = \tilde{g} \circ \tilde{f}$ . We define a compatible system  $\{\tilde{h}_{U\mathcal{U}'}, \theta\}$  of  $\tilde{h}$  by taking the compositions of  $\{\tilde{f}_{U\mathcal{U}'}, \tau\}$  and  $\{\tilde{g}_{U_1\mathcal{U}'_1}, \lambda\}$ , in which the injection for each pair  $U'_{p,i} \in \mathcal{U}' \subset U_{f(p),j} \in \mathcal{U}_1$  is chosen to be the geodesic one, i.e., the one which induces the identity map on the tangent space of  $f(p)$ . □

Now consider a good  $C^\infty$  map  $\tilde{f} : X \rightarrow X'$  with an isomorphism class of compatible systems  $\xi$ . Then we have an isomorphism class of pull-back orbibundles  $(TX')^*_\xi$  over  $X$  and the good  $C^\infty$  map  $\tilde{f}_\xi : (TX')^*_\xi \rightarrow TX'$ . For any  $C^\infty$  section  $\tilde{s}$  of  $(TX')^*_\xi$ , we take the composition  $\tilde{f}_{\xi,s} = \text{Exp} \circ \tilde{f}_\xi \circ \tilde{s}$  from  $X$  into  $X'$ . Then  $\tilde{f}_{\xi,s}$  is a good  $C^\infty$  map with an isomorphism class of compatible systems determined by  $\xi$ . A natural question is: given a good map  $\tilde{g}$  nearby  $\tilde{f}$  with an isomorphism class of compatible systems, is there an isomorphism class of compatible systems  $\xi$  of  $\tilde{f}$ , and a  $C^\infty$  section  $\tilde{s}$  of the pull-back orbifold bundle  $(TX')^*_\xi$  such that  $\tilde{g}$  is realized as  $\tilde{f}_{\xi,s}$ ? If there is, will  $\xi$  and  $\tilde{s}$  be unique? These questions seem to be non-trivial in general, but as we shall see, they can be dealt with in certain special cases, e.g., when  $\tilde{f}, \tilde{g}$  are pseudo-holomorphic maps from a complex orbicurve into an almost complex orbifold.

Suppose  $\tilde{\gamma} : I \rightarrow X$  is a  $C^\infty$  path in  $X$ . Then it is easy to see that  $\tilde{\gamma}$  is a good map so that for each compatible system of  $\tilde{\gamma}$ , and any orbifold bundle  $E$  over  $X$ , there is a pull-back orbifold bundle  $E^*$  over  $I$ , which is a smooth vector bundle on  $I$ . One can verify that when  $\tilde{\gamma}$  is a parametrized geodesic, there is a unique isomorphism class of compatible systems of  $\tilde{\gamma}$ , using the fact that geodesics satisfy the unique continuity property. Therefore all of the pull-back bundles are isomorphic. We consider especially the case when  $E = TX$ . Let  $\nabla$  be a Riemannian connection on  $E$ , and let  $\nabla^*$  be the pull-back connection on  $E^*$ . The parallel transport of  $E^*$  along  $I$ ,  $\text{Par}_{ab} : E^*_a \rightarrow E^*_b$  defined using the connection  $\nabla^*$ , defines through the exponential map a diffeomorphism between  $V_{\gamma(a)}$  and  $V_{\gamma(b)}$ ,

which is equivariant under any automorphism of  $V_{\gamma(a)}$  (and  $V_{\gamma(b)}$ ) which fixes the tangent vector of  $\tilde{\gamma}$ .

**Lemma 4.4.15:** *Let  $(TX')_{\xi,s}^*$  be the pull-back orbifold bundle defined by  $\tilde{f}_{\xi,s}$  with the canonical isomorphism class of compatible systems. Then the map  $\text{Par}_t : (TX')_{\xi}^* \rightarrow (TX')_{\xi,ts}^*$  defined by parallel transportation along parametrized geodesics  $\tilde{\gamma}_x(t) = \text{Exp} \circ \tilde{f}_{\xi} \circ (x, t\tilde{s}(x))$  in  $X'$  is an isomorphism for each  $t \in [0, 1]$ .*

**Proof:** This follows from the fact that parallel transport along geodesics is solely determined by the Riemannian connection on  $TX'$  which is compatible with the transition maps of  $(TX')_{\xi,ts}^*$  for each  $t \in [0, 1]$ .  $\square$

**Definition 4.4.16:** *A sequence of pairs  $(\tilde{f}_n, \xi_n)$  is said to converge to  $(\tilde{f}_0, \xi_0)$  in the  $C^\infty$  topology if there is a sequence of compatible systems  $\{\tilde{f}_{n,UU'_n}, \lambda_n\}$  of  $\tilde{f}_n$  representing  $\xi_n$  and a compatible system  $\{\tilde{f}_{0,UU'_0}, \lambda_0\}$  of  $\tilde{f}_0$  representing  $\xi_0$  with the following property: for each  $U \in \mathcal{U}$ , there is an integer  $n(U) > 0$  such that for each  $n \geq n(U)$ , there is an injection  $\delta_{U'_n}$  from the uniformizing system of  $U'_n$  into that of  $U'_0$ , such that  $\delta_{U'_n} \circ \tilde{f}_{n,UU'_n}$  converges to  $\tilde{f}_{0,UU'_0}$  in  $C^\infty$ , and  $\delta_{U'_{1,n}} \circ \lambda_n(i) = \lambda_0(i) \circ \delta_{U'_{2,n}}$  holds for  $n \geq \max(n(U_1), n(U_2))$  for any injection  $i$  of inclusion  $U_2 \rightarrow U_1$ .*

Recall that in the smooth case, the pull-back bundle can be constructed by fiber product. Namely, if  $p : E \rightarrow Y$  is a bundle and  $f : X \rightarrow Y$  is a map,  $f^*E = \{(x, v); f(x) = p(v)\}$ . The latter is often denoted by  $X \times_Y E$ . In the orbifold case,  $f^*E$  is not the ordinary fiber product. However, one can define an orbifold fiber product as follows.

Suppose that  $f_1 : X \rightarrow Z, f_2 : X_2 \rightarrow Z$  are good maps. Let  $(\tilde{f}_{W_1W}^1, \lambda_{W_1W}^1), (\tilde{f}_{W_2W}^2, \lambda_{W_2W}^2)$  be compatible systems for the same cover  $\{W\}$  of  $Z$ . Then, the orbifold fiber product (still denoted by  $X_1 \times_Z X_2$ ) is constructed by gluing the  $(W_1 \times_W W_2, G_1 \times_G G_2)$  together, where  $G_i$  (resp.  $G$ ) are the local group of  $X_i$  (resp.  $Z$ ). If  $W_1 \times_Z W_2$  is smooth,  $X_1 \times_Z X_2$  is an orbifold. It is obvious that  $f^*E = X \times_Y E$ . One can check that the projections  $W_1 \times_W W_2 \rightarrow W_1, W_2$  define good maps  $p_i : X_1 \times_Z X_2 \rightarrow X_1, X_2$ .

## 4.5 A canonical stratification of orbifolds

In this subsection, we give a brief discussion on the structure of the singular set  $\Sigma X$  and describe a canonical stratification of orbifolds. See [K1] for more details.

The singular set  $\Sigma X$  of an orbifold  $X$  is not an orbifold in general. But we can consider  $\Sigma X$  as an immersed image of a disjoint union of orbifolds. More precisely, let  $(1) = (H_p^0), (H_p^1), \dots, (H_p^{n_p})$  be all the orbit types of a geodesic chart  $(V_p, G_p, \pi_p)$  at  $p$ . For  $q \in U_p = \pi_p(V_p)$ , we may take  $U_q$  small enough so that  $U_q \subset U_p$ . Then any injection  $\phi : V_q \rightarrow V_p$  induces a unique homomorphism  $\lambda_\phi : G_q \rightarrow G_p$ , which gives a correspondence  $(H_q^i) \rightarrow \lambda_\phi(H_q^i) = (H_p^j)$ . This correspondence is independent of the choice of  $\phi$ . Consider the set of pairs:

$$\widetilde{\Sigma U}_p = \{(q, (H_q^i)) | q \in \Sigma U_p, i \neq 0\}.$$

Take one representative  $H_q^i \in (H_q^i)$ . Then the pair  $(q, (H_q^i))$  determines exactly one orbit  $[\tilde{q}]$  in the fixed point set  $V_p^{H_p^j}$  by the action of the normalizer  $N_{G_p}(H_p^j)$ , where  $\tilde{q} = \phi(q)$ ,  $(H_p^j) = \lambda_\phi(H_q^i)$ . The correspondence  $(q, (H_q^i)) \rightarrow [\tilde{q}]$  gives a homeomorphism

$$\widetilde{\Sigma U}_p \cong \coprod_{j=1}^{n_p} V_p^{H_p^j} / N_{G_p}(H_p^j), \text{ (disjoint union),}$$

which gives  $\widetilde{\Sigma X} = \{(p, (H_p^j)) | p \in \Sigma X, j \neq 0\}$  an orbifold structure

$$\{\pi_{p,j} : (V_p^{H_p^j}, N_{G_p}(H_p^j)) \rightarrow V_p^{H_p^j}/N_{G_p}(H_p^j) : p \in X, j = 1, \dots, n_p\}.$$

The canonical map  $\pi : \widetilde{\Sigma X} \rightarrow \Sigma X$  defined by  $(p, (H_p^j)) \rightarrow p$  is surjective, which has a  $C^\infty$  lifting  $\tilde{\pi}$  given locally by embeddings  $V_p^{H_p^j} \rightarrow V_p$ . Therefore  $\widetilde{\Sigma X}$  can be regarded as a resolution of the singular set  $\Sigma X$ , called the canonical resolution.

A point  $(p, (H_p^j))$  in  $\widetilde{\Sigma X}$  is called generic if  $G_p = H_p^j$ . The set  $\widetilde{\Sigma X}_{gen}$  of all generic points is open dense in  $\widetilde{\Sigma X}$ , and the map  $\pi|_{\widetilde{\Sigma X}_{gen}} : \widetilde{\Sigma X}_{gen} \rightarrow \Sigma X$  is bijective. Hence we have a partition of  $X$  into a disjoint union of smooth manifolds:

$$X = X_{reg} \cup \widetilde{\Sigma X}_{gen},$$

which is called the canonical stratification of  $X$ .

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